The Spectrum of Random Permutation Matrices

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Abstract

This thesis contains mainly two parts. In the first part, we establish the limit laws of linear statistics of random permutation matrices. In the second part, we show Central Limit theorems for the characteristic polynomial of random permutation matrices and multiplicative class functions. A third part contains some natural extensions of the techniques of part one and two.

Zusammenfassung

Diese Dissertation besteht aus zwei Hauptteilen. Im ersten Teil werden wir die Verteilungen im Limit Linearer Statistiken von Zufallsmatrizen basierend auf Permutationen behandeln. Der zweite Teil besteht aus Zentralen Grenzwertsätzen für das Charakteristische Polynom ebensolcher Zufallsmatrizen, welche auf Permutationen basieren. In einem dritten Teil beweisen wir zwei Theoreme anhand derselben Methoden in Teil eins. Betrachtungen im dritten Teil sind eine natürliche Fortsetzung des ersten und zweiten Teils.

To my grandparents

Contents

Chapter 1

Introduction

The study of eigenvalue statistics of random matrices is the core of Random Matrix Theory (RMT). Eigenvalues of random matrices are known to model in a numerical or rigorous way the statistical properties of the observed spectra of complex nuclei in chaotic quantum systems and the statistical zeros of the Riemann Zeta function.

Random matrices were first introduced in the 30's by Hsu, Wishart and others (see [55]). The study of their properties in connection with nuclear physics began in the mid 50's by Wigner [79], [80], [81], who introduced the concept of statistical distribution of nuclear energy levels and ensembles of random matrices. In a series of papers [29], [30], [31], [32], [33], Dyson established a classification of invariant matrix ensembles by their invariance properties under time reversal. This classification can be seen as the foundation of RMT and one refers to these ensembles often as Dyson-Wigner Ensembles or Classical Matrix Ensembles. Early results on RMT are given in [61], a collection of important papers, or in the monograph of Mehta [55]. We refer for a broader survey on early developments in RMT to [37].
The main motivation of RMT was to obtain a better understanding of the statistical behavior of the energy of heavy nuclei in chaotic quantum systems. It is well-known today that agreements between observations of complex nuclei and random matrices can be found in terms of level spacings, pair correlation or variance by interpreting the eigenvalues as the high energy. For example, eigenvalues of matrices in the matrix ensembles introduced in [79] model the spectra of complex nuclei in an exact way.
The mathematical society, in particular number theorists, became more interested in RMT, when Montgomery and Dyson met at a conference in 1972 and discovered the surprising fact that the pair correlation of the zeros of the Riemann Zeta function, studied by Montgomery [56], are conjectured to behave similarly as the pair correlation of eigenvalues from matrices in Dyson's Circular Ensembles. Montgomery's conjecture was numerically confirmed by Odlyzko in 1987 [58]. Since then, the study of the Riemann Zeta function and prime numbers by using probabilistic approaches became an active field (see for instance of [39], [48], [49], [64], [65]). The work of Keating and Snaith in 2000 [49] shows the statistical relationship between the moments of characteristic polynomials of matrices from Dyson-Wigner Ensembles and moments of the Riemann Zeta Function and families of L-functions. This led to further research of characteristic polynomials of random matrices such as in [13], [22], [44], [49], [40] or [82].
Most of the interest in RMT concentrates on fluctuations of the spectrum of ran-

dom matrices. An important property is the universal behavior of local and global fluctuations. For example, eigenvalue correlation scaled on the average level spacing do not depend on the probability distribution (see for instance [60]), the limit distribution of largest, respectively smallest eigenvalues is non-random (see for example [73]), linear statistics (see for example [24], [17], [68]) and the characteristic polynomial satisfy in general a CLT (see for instance [19], [49], [40], [82]).
In this thesis, we show some results on global fluctuations of random permutation matrices and matrix groups closely related to random permutation matrices. We study global fluctuations in terms of linear statistics and in terms of the characteristic polynomial. To present these results in a broader context, we will now recall the classical matrix ensembles.

1.1 Classical Invariant Matrix Ensembles

The underlying structure in RMT are the matrix models. A matrix model is a probability space, where the sample space is a set of matrices. The probability measure on the sample space can admit invariance properties, in which case the matrix model is called an invariant matrix model. Otherwise, it is called a non-invariant matrix model. In a series of papers in the 60's, Dyson classified some invariant matrix models by their invariance properties, known today as the Gaussian Wigner Ensembles GUE, GOE and GSE and Dyson's Circular Ensembles CUE, COE and CSE [29], [30], [31], [32], [33]. They are the most common (invariant) matrix models in RMT and are also referred to as the classical (invariant) matrix ensembles.

1.1.1 Gaussian Ensembles

1. The Gaussian Unitary Ensemble (GUE), $\beta = 2$.
   Let $H = (h_{ij})_{1\le i,j\le N}$ be a N-dimensional Hermitian matrix. The GUE is given by the sample set of N-dimensional Hermitian matrices H and the probability measure
   $$dP(H) = \frac{1}{Z_N} \exp\left(-\mathrm{Tr}(H^2)\right) dH, \tag{1.1.1}$$
   where
   $$Z_N = \int \exp\left(-\mathrm{Tr}(H^2)\right) dH \tag{1.1.2}$$
   denotes the so-called partition function which normalizes the measure dP so that it becomes a probability measure. dH is the Lebesgue measure on the entries, i.e.
   $$dH = \prod_{1\le i<j\le N} d\mathrm{Re}(h_{ij}) d\mathrm{Im}(h_{ij}) \prod_{i=1}^{N} dh_{ii}. \tag{1.1.3}$$
   The matrix elements of H, where H is chosen with respect to dP, are independent up to symmetry and Gaussian distributed. In fact, the entries h_{ij}

are given by two independent families of i.i.d. Gaussian random variables $\{h_{ij} : 1 \le i < j \le N\}$ and $\{h_{ii} : 1 \le i \le N\}$, so that $\mathbb{E}[h_{ij}] = 0$ and (by the right normalization) $2\mathbb{E}\left[|h_{ij}|^2\right] = \mathbb{E}\left[h_{11}^2\right] = 1$.

The measure dP is invariant under conjugation by unitary matrices, i.e.

$$dP(H) = dP(UHU^*), \tag{1.1.4}$$

so that for diagonalizable matrices $H = U\Lambda U^*$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$, it is clear that dP can be expressed in terms of the eigenvalues $\lambda_1, \dots, \lambda_N$ by the joint distribution

$$d\rho_{\mathrm{GUE}}(\lambda_1, \dots, \lambda_N) = \frac{1}{C_N} \exp\left(-\sum_{i=1}^{N} \lambda_i^2\right) \triangle_N(\lambda) \prod_{i=1}^{N} d\lambda_i, \tag{1.1.5}$$

where

$$C_N^{-1} = (2\pi)^{-N/2} 2^{N^2/2} \prod_{j=1}^{N} \frac{\Gamma(2)}{\Gamma(1+j)} = (2\pi)^{-N/2} 2^{N^2/2} \prod_{j=1}^{N} \frac{1}{j!} \tag{1.1.6}$$

is the partition function. The Jacobian for the change of variables is the Vandermonde Determinant $\triangle_N(\lambda)$ defined by

$$\triangle_N(\lambda) = \prod_{1 \le i < j \le N} |\lambda_j - \lambda_i|^2. \tag{1.1.7}$$

Sometimes, $\triangle_N(\lambda)$ is written within the exponential and

$$\sum_{i=1}^{N} \lambda_i^2 - \sum_{i \ne j}^{N} \log|\lambda_i - \lambda_j|^2 \tag{1.1.8}$$

is called the total potential energy. This comes from the physical meaning of the Gibbs measure (1.1.5): it gives the distribution of energy levels, where the particles (eigenvalues) on the line are interacting via the repulsive logarithmic Coulomb potential

$$V(R) = -\log R \tag{1.1.9}$$

at inverse temperature $\beta = 2$ (i.e. at temperature $(K\beta)^{-1}$, where K is the so called Boltzmann constant) [17]. The following two matrix ensembles are corresponding to the inverse temperatures $\beta = 1$ and $\beta = 4$.

2. The Gaussian Orthogonal Ensemble (GOE), $\beta = 1$.
The GOE is given by the set of real symmetric N-dimensional matrices $H = (h_{ij})_{1 \le i,j \le N}$ and the probability measure is given by

$$dP_1(H) = \frac{1}{Z_1} \exp\left(-\frac{1}{2}\mathrm{Tr}(H^2)\right) dH, \tag{1.1.10}$$

where

$$Z_1 = \int \exp\left(-\frac{1}{2}\mathrm{Tr}(H^2)\right) dH \tag{1.1.11}$$

is the partition function and

$$dH = \prod_{1\le i\le j\le N} dh_{ij}. \tag{1.1.12}$$

Given a H chosen with respect to dP_1 on the group of real symmetric N-dimensional matrices, the matrix elements are independent Gaussian random variables. More precisely, the diagonal entries are i.i.d. $\mathcal{N}(0,1)$ random variables and the off diagonal entries are, up to symmetry, i.i.d. $\mathcal{N}(0,1/2)$ random variables (by the right normalization). The measure dP_1 is invariant under orthogonal conjugation, which gives the matrix model its name. Again, by the invariance property, dP_1 can be written in terms of the eigenvalues $\lambda_1,\ldots,\lambda_N$. After a change of variables, dP_1 becomes

$$d\rho_\beta(\lambda) = C_\beta^{-1}\exp\left(-\frac{1}{2}\beta\sum_{i=1}^N \lambda_i^2\right)\prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta\prod_{i=1}^N d\lambda_i, \tag{1.1.13}$$

where

$$C_\beta^{-1} = (2\pi)^{-N/2}\beta^{N/2+\beta N(N-1)/4}\prod_{j=1}^N\frac{\Gamma(1+\beta/2)}{\Gamma(1+j\beta/2)}. \tag{1.1.14}$$

The joint probability density of the eigenvalues from the GOE is explicitly given by setting $\beta=1$. Note that for $\beta=2$, $\rho_2(\lambda)$ agrees with the joint probability density of eigenvalues from the GUE.

3. The Gaussian Symplectic Ensemble (GSE), $\beta=4$.
   By setting $\beta=4$, similar formulas can be derived for the GSE on the sample set of real quaternion self-dual matrices. We refer for further inquiries to [55].

Of course, these are not the only invariant matrix models in RMT, as the following example shows. Let $\mathcal{H}_N$ be the set of N-dimensional Hermitian matrices. The measure dP given by

$$dP(H) = \frac{1}{Z_N}\exp(-M\mathrm{Tr}V(H))dH \tag{1.1.15}$$

is always invariant under conjugation by unitary matrices. Here V denotes a polynomial of degree 2ℓ, $\ell\ge 1$, with positive leading coefficient. Whenever V is quadratic, the matrix elements of H are independent.

$$Z_N = \int \exp(-M\mathrm{Tr}V(H))dH \tag{1.1.16}$$

denotes the partition function and normalizes the measure dP so that it becomes a probability measure. The set $\mathcal{H}_N$ endowed with the measure given in (1.1.15) is an example of a more general class of invariant matrix models.

Wigner studied in the 50's Hermitian and symmetric matrices, where all entries are independent (up to symmetry), the diagonal entries are i.i.d. and the off diagonal entries are centered, with the same finite variance [79]. Except from the examples given above, these matrices define in general non-invariant measures and the matrices are called Wigner matrices.

1.1.2 Circular Ensembles

In literature, computing various statistics about the eigenvalues of matrices from the GUE, GOE and GSE can usually applied to other matrix ensembles with different underlying symmetries. For example, results on the GUE, GOE and GSE apply in general to Dyson's Circular Ensembles, namely the corresponding CUE, COE and CSE.

1. The Circular Unitary Ensemble (CUE), $\beta = 2$.
   The sample set of the CUE is the group of N-dimensional unitary matrices $U = (u_{ij})_{1\le i,j\le N}$. The probability measure is the (unique) normalized Haar measure $d\mu_{\mathrm{CUE}}$, which is invariant under unitary transformation
   $$U \to SUV, \tag{1.1.17}$$
   for any S and V being $N \times N$ unitary matrices. The normalization is so that $d\mu_{\mathrm{CUE}}$ defines a probability measure. The existence of the Haar measure is clear, since the group of unitary matrices is a compact Lie group.

   For eigenvalues $\lambda_1 = e^{2i\pi\varphi_1}, \dots, \lambda_N = e^{2i\pi\varphi_N}$ with eigenangles $\varphi_1, \dots, \varphi_N \in [0,1]$, Weyl's integration formula shows immediately the equality of the Haar measure and the joint probability density of eigenvalues of N-dimensional matrices $U \in \mathcal{U}_N$ from the CUE by
   $$\int_{\mathcal{U}_N} f(U) d\mu_{\mathrm{CUE}} = \frac{1}{(2\pi)^N N!} \int_{[0,1]^N} f(\lambda_1, \dots, \lambda_N) |\triangle(\lambda)|^2 d\varphi_1 \dots d\varphi_N. \tag{1.1.18}$$
   Here, f denotes a class function and $|\triangle(\lambda)|$ is the Vandermonde Determinant given by
   $$\prod_{1\le i<j\le N} |\lambda_i - \lambda_j| . \tag{1.1.19}$$
   The square in $|\triangle(\lambda)|^2$ refers to the inverse temperature $\beta = 2$. Note that class functions are functions f, so that $f(U) = f(\lambda_1, \dots, \lambda_N)$ are symmetric in all of its variables.

2. The Circular Orthogonal Ensemble (COE), $\beta = 1$.
   The sample space of the COE is the group of N-dimensional symmetric unitary

matrices O. The probability measure is the (unique) normalized Haar measure, which is invariant under unitary transformation

$$O \to W^T O W, \tag{1.1.20}$$

for any W being a $N \times N$ unitary matrix. Alternatively, if U is a CUE matrix, then $U^T U$ is from the COE. Again, by Weyl's integration formula, we find the joint probability density of the eigenvalues $\lambda_1 = e^{2i\pi\varphi_1}, \ldots, \lambda_N = e^{2i\pi\varphi_N}$, for $\varphi_1, \ldots, \varphi_N \in [0,1]$ and f being a class function, by

$$\frac{1}{(2\pi)^N Z_\beta} \int_{[0,1]^N} f(\lambda_1, \ldots, \lambda_N) \prod_{1 \le i < j \le N} |\triangle(\lambda)|^\beta d\varphi_1 \ldots d\varphi_N. \tag{1.1.21}$$

The partition function Z_β is given by

$$Z_\beta = \frac{\Gamma(1/2\beta N + 1)}{\Gamma(1/2\beta + 1)^N}. \tag{1.1.22}$$

Setting $\beta = 1$ gives explicitly the joint probability density function for the eigenvalues of matrices from the COE.

3. The Circular Symplectic Ensemble (CSE), $\beta = 4$.
The sample space of the CSE is given by self dual quaternion unitary matrices. The joint probability density function for the CSE can be derived by setting $\beta = 4$ in (1.1.21). We refer for more details to [55].

It should be clear from the above discussed joint probability distribution of the eigenvalues for the Gaussian and the Circular Ensembles that the eigenvalues are a.s. distinct. Note that this does not apply to the set of random permutation matrices under the uniform measure or under the Ewens distribution.

1.1.3 Universal Behavior of Eigenvalue Statistics

In RMT, one usually distinguish the study of global and local statistics of eigenvalues. Global statistics are mostly studied in terms of fluctuations of linear statistics, the characteristic polynomial or empirical spectral measures. We give some results on global fluctuations in a brief survey below:

1. Let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of a $N \times N$ random matrix. The empirical spectral measure of the renormalized eigenvalues is defined by

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i/\sqrt{N}}. \tag{1.1.23}$$

One of the earliest results in RMT is Wigner's famous theorem of the semicircle law [79] for Wigner matrices $H = (h_{ij})_{1 \le ij \le N}$, i.e. for Hermitian or symmetric matrices with independent entries (up to symmetry) so that the diagonal entries have the same distribution and the off diagonal entries are centered with the same mean square deviation.

Theorem 1.1.1 (Wigner's Theorem). *Assume that* $\mathbb{E}\left[|h_{12}|^2\right] = 1$ *and* $\mathbb{E}\left[h_{11}^2\right] < \infty$, *then*

$$\lim_{N\to\infty} dL_N(x) = d\sigma(x), \tag{1.1.24}$$

in probability, where

$$d\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}. \tag{1.1.25}$$

This theorem is universal, i.e. it holds for general matrices satisfying overall symmetry requirements and it does not depend on further details of the distributions of the matrix entries. In other words, Wigner's Theorem holds for non-invariant Wigner matrix ensembles as well as for invariant Hermitian matrix ensembles such as the Gaussian ensembles. Details of the distributions of the matrix entries are washed away in the large scale limit.
For the Circular Ensembles, the so-called equilibrium measure is the uniform measure.

2. The linear statistic for a $N \times N$ matrix M is defined by

$$I_N(f) = \sum_{i=1}^{N} f(\lambda_i), \tag{1.1.26}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues and f is a function on the eigenvalues. Linear statistics for invariant ensembles have been studied for instance by Costin and Lebowitz [17], Diaconis and Evans [24], Diaconis and Shahshahani [25], Dumitriu and Edelman [27], Johansson [45], [46], Johnson [47], Pastur [59], Soshnikov [68] or Wieand [77]. Linear statistics for non-invariant ensembles have been studied for example by Bai [6], Bai and Silverstein [8], Boutet de Monvel, Pastur and Shcherbina [21], Chatterjee [16], Dumitriu and Soumik [28], Khorunzhy, Khoruzhenko and Pastur [51], Lytova and Pastur [54], Sinai and Sohsnikov [66] or Wieand for random permutation matrices [76].

All previous results share two features:

- The variance of linear statistics does not diverge for smooth enough functions f and so, there is no normalization needed to get a limit law. For less smooth functions the variance blows up logarithmically.
- Those fluctuations are asymptotically Gaussian, except in [59], where invariant ensembles with more than one cut are shown to have non-Gaussian fluctuations.

We recapture here briefly some results for invariant matrix models. In [17], Costin and Lebowitz study the linear statistic of matrices from the GUE, GOE and GSE, where f is the indicator function of the eigenvalues in an interval on the real line with given length. For the circular ensembles CUE, COE and CSE, Wieand studies the linear statistic for f being the indicator function of the eigenvalues in a given arc on the unit circle [77].

Diaconis and Evans [24], Johansson [46] and Soshnikov [68] study linear statistics of matrices from the CUE (extending it also to COE and CSE) for general functions f. In [24], they use calculations of joint moments of traces, obtained by Diaconis and Shahshahani [25]. They recover the results from [17] and [77]. Furthermore, they recover results for the logarithm of characteristic polynomials for unitary matrices from [44] and [49] by using the principle of the argument (see Section 7 in [24]). Their work defer from Johansson [46] in the sense that they do not need the strong Szegö theorem or Toeplitz determinants. Johansson makes use of these for deriving explicit rates of convergences. Soshnikov [68] studies global and local linear statistics where f is a trigonometric polynomial, by using combinatorial methods.

Permutation matrices are a subgroup of unitary matrices, but linear statistics for random permutation matrices show a different behavior. In Chapter 2, we will study linear statistics for a non-invariant model of random permutation matrices. An overview of the obtained results is given in Section 1.2.

3. The characteristic polynomial of a $N \times N$ unitary M can be studied for any $x = e^{2i\pi\varphi}$, $0 \leq \varphi < 1$, by the function

$$Z_M(x) = \left(-\frac{1}{x}\right)^N \det(M - xI), \tag{1.1.27}$$

which has the same zeros as the characteristic polynomial of M. Keating and Snaith prove in [49] a Central Limit Theorem result for $\log Z_M(x)$, where M is from the CUE. They show that the imaginary and the real part of

$$\frac{\log Z_M(x)}{\sqrt{(1/2)\log N}} \tag{1.1.28}$$

converges in law to a bivariate standard normal distribution. Hughes, Keating and O'Connell refined this result in [44] by giving a CLT for $\log Z_M(x)$, evaluated at finitely many distinct points on the circle. In [40], Hambly, Keevash, O'Connell and Stark give a Gaussian limit for the logarithm of the characteristic polynomial of random permutation matrices P under uniform measure on the symmetric group S_N. In [83], Zeindler extended the result of [40] to the Ewens distribution on the symmetric group and to the logarithm of multiplicative class functions, introduced in [22].

We will study the characteristic polynomial of random permutation matrices in Chapter 3. An overview of the obtained results is given in Section 1.3.

The study of the local statistics concentrates in general on eigenvalue spacings, largest or smallest eigenvalues or the joint distribution of the eigenvalues in a interval of given length of order $1/N$. In general, the eigenvalues are regularly spaced, pairs are close and gaps are rare, due to level repulsion (see for example [35]).

In case of Hermitian or symmetric matrices, the observation that bulk statistics differ vastly from edge statistics is substantial, i.e. the behavior of the eigenvalues on the support of the semicircular law differs from the behavior of the eigenvalues near the edges of the support (see for instance [72] or [73]). We mention briefly some results on local statistics.

1. With appropriate normalization, the bulk of the spectrum of Wigner matrices lies on the interval $(-2, 2)$ (see [35], [70], [71] or [72]) and the density of the eigenvalues follow the semicircle law mentioned above. Moreover, the largest eigenvalue sticks to the bulk (see for instance [38] for GUE).

2. The largest eigenvalue follows the so-called Tracy-Widom distribution [67], [72]:

   Theorem 1.1.2 (Tracy-Widom Theorem). *Let*

   $$F_{\beta,N}(t) = \mathbb{P}(\lambda_{max} < t), \tag{1.1.29}$$

   be the distribution function of the largest eigenvalue of a $N \times N$ matrix from the GOE, GUE or GSE, corresponding to $\beta = 1, 2, 4$. Then for σ being the variance of the off-diagonal entries,

   $$\lim_{N\to\infty} F_{\beta,N}\left(2\sigma\sqrt{N} + \frac{t\sigma}{N^{1/6}}\right) = F_\beta(t) \tag{1.1.30}$$

   exists and is given for $\beta = 1, 2$ or 4 explicitly.

   The limit distribution of the largest eigenvalue is called Tracy-Widom distribution. For $\beta = 2$,

   $$\lim_{N\to\infty} F_{2,N}\left(2\sigma\sqrt{N} + \frac{t\sigma}{N^{1/6}}\right) = \exp\left(-\int_{-t}^{\infty}(y-t)q^2(y)dy\right), \tag{1.1.31}$$

   where q is the unique solution to the Painlevé II equation

   $$\frac{d^2q}{dx^2} = xq + 2q^3 \tag{1.1.32}$$

   with boundary conditions

   $$q(t) \sim Ai(t) \quad t \to \infty. \tag{1.1.33}$$

   In [67], Soshinikov proved for (non-invariant) Wigner matrices that under the condition that the distribution of the off diagonal entries is symmetric,

   $$\lim_{N\to\infty} \mathbb{P}(\lambda_{\max} \leq 1 + \frac{t}{2N^{2/3}}) = F_\beta(t), \tag{1.1.34}$$

   where $\beta = 1$ if the matrices are real symmetric and $\beta = 2$ if the matrices are Hermitian. Thus, he established the universality of the Tracy-Widom distribution.

3. Eigenvalues are in the limit localized (See for instance [1], [71] or [70].)

We will not pursue the study of the local behavior of eigenvalues in this thesis, but rather concentrate on the global behavior in terms of linear statistics and the characteristic polynomial. We continue by outlining the content of Chapter 2, Chapter 3 and Chapter 4.

1.2 Overview of Chapter 2

In Chapter 2, we study linear statistics of random permutation matrices for general functions f. The results are presented in the same way as in the paper "On Fluctuations of Eigenvalues of Random Permutation Matrices", a joint work with Gérard Ben Arous [10]. We show that the behavior of the variance of linear statistics of random permutation matrices follow the general pattern, i.e. the variance does not diverge for smooth enough functions f and for less smooth functions it blows up logarithmically. But we will also prove that the asymptotic limit law is not Gaussian but infinitely divisible when the function is smooth enough. This is in contrast to the case where the function is less regular. The fluctuations are then indeed asymptotically Gaussian after normalization. This Gaussian behavior for random permutation matrices was previously proved by K. Wieand (see [76]) for the special case where the linear statistic is the number of eigenvalues in a given arc, and for uniformly distributed permutations.
In order to outline the main results of Chapter 2, we introduce some notations.

For N being an integer, let $\mathcal{S}_N$ denote the symmetric group. We consider permutations $\sigma \in \mathcal{S}_N$ sampled under the Ewens measure or Ewens distribution defined by

$$\nu_{N,\theta}(\sigma) = \frac{\theta^{K(\sigma)}}{\theta(\theta+1)\dots(\theta+N-1)}, \tag{1.2.1}$$

where $\theta > 0$ and $K(\sigma)$ is the total number of cycles of the permutation σ.
The Ewens measure or Ewens distribution is a well-known measure on the the symmetric group $\mathcal{S}_N$, appearing for example in population genetics [36], [75]. It can be viewed as a generalization of the uniform distribution ($\theta = 1$) and has an additional weight depending on the total number of cycles.
We denote by M_σ the permutation matrix defined by the permutation $\sigma \in \mathcal{S}_N$. For any $1 \leq i, j \leq N$, the entry $M_\sigma(i,j)$ is given by

$$M_\sigma(i,j) = \mathbb{1}_{i=\sigma(j)}. \tag{1.2.2}$$

M_σ is unitary and its eigenvalues belong to the unit circle $\mathbb{T}$. We denote the eigenvalues by

$$\lambda_1(\sigma) = e^{2i\pi\varphi_1(\sigma)}, \dots, \lambda_N(\sigma) = e^{2i\pi\varphi_N(\sigma)} \in \mathbb{T}, \tag{1.2.3}$$

where $\varphi_1(\sigma), \dots, \varphi_N(\sigma)$ are in $[0,1]$.

For any real-valued periodic function f of period 1, we study the linear statistic

$$I_{\sigma,N}(f) = \mathrm{Tr}\widetilde{f}(M_\sigma) = \sum_{i=1}^{N} f(\varphi_i(\sigma)), \tag{1.2.4}$$

where $\widetilde{f}(e^{2i\pi\varphi}) = f(\varphi)$ is a function on the unit circle $\mathbb{T}$.
The limit law of the linear statistic depends strongly on the smoothness of f. Due to the fact that the spectrum of the permutation matrix M_σ is easily expressed in terms of the cycle counts of the random permutation σ, i.e. the numbers $\alpha_j(\sigma)$ of cycles of length j $(1 \le j \le N)$, we will be able to quantify the smoothness of f by the error made in the composite trapezoidal approximation to the integral of f [20]. More precisely, the smoothness of f is expressed in terms of the sequence $(R_j(f))_{1\le j\le N}$ given by

$$R_j(f) = \frac{1}{j}\sum_{k=0}^{j-1} f\left(\frac{k}{j}\right) - \int_0^1 f(x)dx. \tag{1.2.5}$$

Recall the definition of a function being of bounded variation.

Definition 1.2.1. *A real-valued function f defined on an interval $[a, b]$ is said to be of bounded variation if*

$$\sup_P \sum_i |f(x_{i+1}) - f(x_i))| < \infty, \tag{1.2.6}$$

where the supremum is taken over all possible partitions $P = \{x_1, x_2, \dots, x_N : x_1 < x_2 < \cdots < x_N\}$, $N \in \mathbb{N}$.

The main results of Chapter 2 are given by the following two theorems.

Theorem 1.2.2. *Let θ be any positive number, and f be a function of bounded variation. Assume that*

$$\sum_{j=1}^{\infty} jR_j(f)^2 \in (0, \infty). \tag{1.2.7}$$

Then,

1. *under the Ewens distribution $\nu_{N,\theta}$, the distribution of the centered linear statistic*
$$I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]$$
converges weakly, as N goes to infinity, to a non-Gaussian infinitely divisible distribution $\mu_{f,\theta}$.

2. *The distribution $\mu_{f,\theta}$ is defined by its Fourier transform*

$$\widehat{\mu}_{f,\theta}(t) = \exp\left(\theta\int (e^{itx} - 1 - itx)dM_f(x)\right), \tag{1.2.8}$$

where the Lévy measure M_f is given by

$$M_f = \sum_{j=1}^{\infty} \frac{1}{j} \delta_{jR_j(f)}. \tag{1.2.9}$$

3. *The asymptotic behavior of the expectation of the linear statistic is given by*

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N \int_0^1 f(x)dx + \theta \sum_{j=1}^{N} R_j(f) + o(1). \tag{1.2.10}$$

Here, the second term $\sum_{j=1}^{N} R_j(f)$ may diverge, but not faster than logarithmically.

$$\sum_{j=1}^{N} R_j(f) = O(\sqrt{\log N}) \tag{1.2.11}$$

4. *The asymptotic behavior of the variance of the linear statistic is given by*

$$\mathrm{Var}[I_{\sigma,N}(f)] = \theta \sum_{j=1}^{N} jR_j(f)^2 + o(1). \tag{1.2.12}$$

Theorem 1.2.3. *Let θ be any positive number, and f be a bounded variation function such that*

$$\sum_{j=1}^{\infty} jR_j(f)^2 = \infty. \tag{1.2.13}$$

Then,

1. *under the Ewens distribution $\nu_{N,\theta}$, the distribution of the centered and normalized linear statistic*

$$\frac{I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]}{\sqrt{\mathrm{Var}\, I_{\sigma,N}(f)}} \tag{1.2.14}$$

converges weakly, as N goes to infinity, to the Gaussian standard distribution $\mathcal{N}(0,1)$.

2. *The asymptotic behavior of the expectation of the linear statistic is given by*

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N \int_0^1 f(x)dx + \theta \sum_{j=1}^{N} R_j(f) + O(1). \tag{1.2.15}$$

Here, the second term $\sum_{j=1}^{N} R_j(f)$ may diverge, but not faster than logarithmically.

$$\sum_{j=1}^{N} R_j(f) = O(\log N) \tag{1.2.16}$$

3. The asymptotic behavior of the variance of the linear statistic is given by

$$\mathrm{Var}[I_{\sigma,N}(f)] \sim \theta \sum_{j=1}^{N} j R_j(f)^2. \tag{1.2.17}$$

In fact, it is not necessary for f being of bounded variation. The best possible assumptions needed involve the notion of Cesaro means of fractional order, which will be introduced in Section 2.2. The theorems with weaker assumptions than given above are also given in Section 2.2. In Section 2.3, we prove corollaries following from the first theorem by using estimates on the trapezoidal approximation. They show that the smoothness assumption holds for $f \in \mathcal{C}^{1+\alpha}$, where $\alpha > 0$ or for f being in the Sobolev space H^s, $s > 0$. The proofs of our main results of Section 2.2 require the Feller coupling, which we introduce in Section 2.4. This is natural since the linear statistic can be expressed in terms of cycle counts of random permutations. We will improve the known bounds for the approximation given by this coupling (see for example [2] or [9]) and relate these bounds to Cesaro means. We will then be ready to prove in Section 2.5 and Section 2.6 our general results as stated in Section 2.2 and that these results imply the two theorems stated in this introduction. For completeness, we give an explicit expression for the expectation and variance of the linear statistics in Section 2.7.

1.3 Overview of Chapter 3

In Chapter 3, we study the global fluctuations of random permutation matrices in terms of their characteristic polynomial. Moreover, we obtain results for more general matrix models, strongly associated to permutation matrices. Let M be a $(n \times n)$-matrix and write for $x = e^{2i\pi\varphi}$, $0 \leq \varphi < 1$,

$$Z_M(x) = \left(-\frac{1}{x}\right)^n \det(M - xI). \tag{1.3.1}$$

Obviously $Z_M(x)$ has the same zeros as the characteristic polynomial of M. By choosing the branch of logarithm in a suitable way, Keating and Snaith prove in [49] the following Central Limit Theorem result: For $n \times n$ CUE matrices U, the joint distribution of the imaginary and the real part of

$$\frac{\log Z_U(x)}{\sqrt{(1/2)\log n}} \tag{1.3.2}$$

converges in law to a bivariate standard normal distribution. Hughes, Keating and O'Connell refined this result in [44]: For a finite set of distinct points $x_1, \dots, x_d$, $\log Z_U(x_i)$ normalized by $\sqrt{(1/2)\log n}$ $(i = 1, \dots, d)$ converges to d i.i.d. standard (complex) normal random variables.

In [40], Hambly, Keevash, O'Connell and Stark give a Gaussian limit for the logarithm of the characteristic polynomial of random permutation matrices P under uniform measure on the symmetric group S_n. Recall that a number φ is said to be of finite type, if the value

$$\eta = \sup_{\gamma} (\liminf n^{\gamma} ||n\varphi|| = 0) \tag{1.3.3}$$

is finite. (Here, n runs over the natural numbers and $||\cdot||$ denotes the distance to the nearest integer.) Hambly, Keevash, O'Connell and Stark show in particular that for irrational φ of finite type, the imaginary and the real part of

$$\frac{\log Z_P(x)}{\sqrt{(\pi^2/12)\log n}} \tag{1.3.4}$$

converge in distribution to standard normal variables. If φ is irrational but not of finite type, $\log Z_P(x)$ needs to be centered (and normalized by the same quantity as above) in order that the imaginary part converges to a standard normal variable. Their result covers the result in [76] by relating the counting function to the imaginary part of $\log Z_P(x)$ (see chapter 4, [40]).
In [83], Zeindler extended the result of [40] to the Ewens distribution on the symmetric group and to the logarithm of multiplicative class functions, introduced in [22].

In Chapter 3, we present the joint work with Dirk Zeindler "The Characteristic Polynomial of a Random Permutation Matrix at Different Points" [19], where we generalize the results in [40] and [83] in two ways. First, we follow the spirit of [44] by considering the behavior of the logarithm of the characteristic polynomial of a random permutation matrix at different points. We will prove the following statement:

Proposition 1.3.1. *Let S_n be endowed with the Ewens distribution with parameter θ and $x_1 = e^{2\pi i\varphi_1}, \dots, x_d = e^{2\pi i\varphi_d} \in \mathbb{T}$ be pairwise of finite type.*
Then for P being a random permutation matrix with permutation in S_n we have, as $n \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \begin{pmatrix} \log(Z_P(x_1)) \\ \vdots \\ \log(Z_P(x_d)) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\mathrm{Re}(N_1), \dots, \mathrm{Re}(N_d), \mathrm{Im}(N_1), \dots, \mathrm{Im}(N_d)$ *independent standard normal distributed random variables.*

Second, we state CLT's for the logarithm of characteristic polynomials for matrix groups related to permutation matrices. In particular, we consider $n \times n$-matrices $M = (M_{ij})_{1 \le i,j \le n}$ of the following form: For a permutation $\sigma \in S_n$ and a complex valued random variable z,

$$M_{ij}(\sigma, z) := z_i \delta_{i,\sigma(j)}, \tag{1.3.5}$$

where z_i is a family of i.i.d. random variables s.t. $z_i \stackrel{d}{=} z$, z_i independent of σ. Here, σ is chosen with respect to the Ewens distribution. Matrices $M_{\sigma,z}$ of the form (1.3.5) can be viewed as generalized permutation matrices $P = M_\sigma = M_{\sigma,1}$, where the 1-entries are replaced by i.i.d. random variables.

We show that under various conditions, the logarithm of the characteristic polynomial of $M_{\sigma,z}$ converges to a complex standard Gaussian distributed random variable after normalization and the behavior at different points is independent in the limit. Moreover, the normalization by $\sqrt{(\pi^2/12)\theta \log n}$, where θ is the parameter of the Ewens distribution, is independent of the random variable z. This covers the result in [40] for the uniform measure and z being deterministic equal to 1. We postpone the precise statements of the main results to Chapter 3, but for $Z_{M_{\sigma,z}}(x) = Z_{n,z}(x)$, they imply the following results:

Proposition 1.3.2. *Let S_n be endowed with the Ewens distribution with parameter θ, z a $\mathbb{T}$-valued random variable and $x \in \mathbb{T}$ be not a root of unity, i.e. $x^m \neq 1$ for all $m \in \mathbb{Z}$.*

Suppose that z is uniformly distributed. Then, as $n \to \infty$,

$$\frac{\operatorname{Re}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \xrightarrow{d} N_R \quad \textit{and} \tag{1.3.6}$$

$$\frac{\operatorname{Im}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \xrightarrow{d} N_I, \tag{1.3.7}$$

with $N_R, N_I \sim \mathcal{N}(0,1)$.

Here, $\operatorname{Re}\left(\log\left(Z_{n,z}(x)\right)\right)$ and $\operatorname{Im}\left(\log\left(Z_{n,z}(x)\right)\right)$ are converging to normal random variables without centering, since the expectation is $o(\sqrt{\log n})$. This will become more clear in the proof (see Section 3.4.1).

Furthermore, the following CLT result holds for $\log Z_{n,z}(x)$, evaluated on a finite set of different points $\{x_1, \ldots, x_d\}$.

Proposition 1.3.3. *Let S_n be endowed with the Ewens distribution with parameter θ, $\overline{z} = (z_1, \ldots, z_d)$ be a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i \varphi_1}, \ldots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \ldots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*

Suppose that $z_1, \ldots, z_d$ are uniformly distributed and independent. Then we have, as $n \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \begin{pmatrix} \log\left(Z_{n,z_1}(x_1)\right) \\ \vdots \\ \log\left(Z_{n,z_d}(x_d)\right) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\operatorname{Re}(N_1), \ldots, \operatorname{Re}(N_d), \operatorname{Im}(N_1), \ldots, \operatorname{Im}(N_d)$ *independent standard normal distributed random variables.*

This shows that the characteristic polynomial of the random matrices $M_{\sigma,z}$ follows the tradition of matrices in the CUE, if evaluated at different points, due to the result by [44].
For the proofs we generalize the methods of [40] and [83] which require the theory of uniformly distributed sequences and the Feller coupling. For the Feller coupling we make use of refined estimates from [10]. This is presented in Section 3.2. In Section 3.3, we state some auxiliary CLT's on the symmetric group, which we will use in Section 3.4 to prove our main results, stated as well in Section 3.4. Chapter 3 is presented in the same way as [19], a joint work with Dirk Zeindler.

1.4 Overview of Chapter 4

In Chapter 4, we present briefly some natural extension of the work on linear statistics and the characteristic polynomial of matrices related to random permutations. $N \times N$-matrices of the form (1.3.5) studied in Chapter 3 were previously introduced by Nikeghbali and Najnudel in [57]. Results on their linear statistics, for both finite and infinite variance, are given in a recent paper by Hughes, Najnudel, Nikeghbali and Zeindler [43]. The results from [43] recover the results by Ben Arous and Dang, presented in Chapter 2. Indeed, by expanding linear statistics for the random matrix model introduced in [57] as in Chapter 2, i.e. expanding linear statistics in terms of the number of cycles, one can see that the techniques of Ben Arous and Dang in [10] apply for the matrix model (1.3.5) of Nikeghbali and Najnudel. We will outline the application of the techniques of [10] for linear statistics in form of (1.3.5) in Chapter 4. To state the theorems of Chapter 4, we will start by introducing some notations.

Let $\sigma \in \mathcal{S}_N$ be a permutation chosen at random from the Ewens distribution with parameter $\theta > 0$. Consider matrices

$$M_{\sigma,z} = (z_i \delta_{i\sigma(j)})_{1 \le i,j \le N}, \tag{1.4.1}$$

where the random variables z_i, $1 \le i \le N$, are i.i.d. random variables with values on the unit circle such that $z_i \overset{d}{=} z$. For a periodic function f on the interval $[0, 1]$ and $f(x) = \tilde{f}(e^{2i\pi x})$, define the linear statistic

$$I_{\sigma,N,z}(f) = \sum_{i=1}^{N} \tilde{f}(\lambda_i) = \sum_{i=1}^{N} f(\varphi_i), \tag{1.4.2}$$

where $\lambda_1 = e^{2i\pi\varphi_1}, \dots \lambda_N = e^{2i\pi\varphi_N}$ are the eigenvalues of $M_{\sigma,z}$. For any cycle of σ of length j, the corresponding j eigenvalues are of the form

$$T_j \cdot \omega, \tag{1.4.3}$$

where

$$\omega^j = 1, \quad T_j = e^{2i\pi\psi_j} \overset{d}{=} \left(\prod_{k=1}^{j} z_k\right)^{1/j}. \tag{1.4.4}$$

Also, for every j we have α_j (the number of j-cycles of σ) different i.i.d. random variables

$$T_j^{(1)}, \ldots, T_j^{(\alpha_j)}. \tag{1.4.5}$$

This gives the $j \cdot \alpha_j$ eigenangles (i.i.d.) for each $1 \leq j \leq N$:

$$\frac{k}{j} + \psi_j^{(1)}, \frac{k}{j} + \psi_j^{(2)}, \ldots, \frac{k}{j} + \psi_j^{(\alpha_j)}, \quad k = 0, \ldots, j-1 \tag{1.4.6}$$

The linear statistic $I_{\sigma,N,z}(f)$ is then equal in law to

$$\sum_{j=1}^{N} \sum_{\ell=1}^{\alpha_j} \left[\sum_{k=0}^{j-1} f(\frac{k}{j} + \psi_j^{(\ell)})\right] = \sum_{j=1}^{N} \sum_{\ell=1}^{\alpha_j} \sum_{k=0}^{j-1} \tau_{j,\ell} f(\frac{k}{j}), \tag{1.4.7}$$

where τ denotes the translation of f by a random factor ψ:

$$\tau_{j,\ell} f(x) = f(x + \psi_j^{(\ell)}). \tag{1.4.8}$$

We introduce the sequence

$$R_{j,\ell}(f) = \frac{1}{j} \sum_{k=0}^{j-1} \tau_{j,\ell} f\left(\frac{k}{j}\right) - \int_0^1 \tau_{j,\ell} f(x) dx = \frac{1}{j} \sum_{k=0}^{j-1} \tau_{j,\ell} f\left(\frac{k}{j}\right) - \int_0^1 f(x) dx, \tag{1.4.9}$$

where f is periodic on $[0,1]$. Clearly, the linear statistic $I_{\sigma,N,z}$ can be expressed in terms of $R_{j,\ell}(f)$ and we have the following generalization of the results from [10].

Theorem 1.4.1. *For $\theta \geq 1$, assume that*

$$\sum_{j=1}^{\infty} j \mathbb{E}\left[R_{j,1}^2(f)\right] < \infty, \tag{1.4.10}$$

then the distribution of

$$I_{\sigma,N,z} - \mathbb{E}\left[I_{\sigma,N,z}\right] \tag{1.4.11}$$

converges to an infinitely divisible distribution μ, defined by

$$\log \hat{\mu} = \sum_{j=1}^{\infty} \frac{\theta}{j} \left(\hat{\nu}_j(t) - 1 - itm_j\right), \tag{1.4.12}$$

where ν_j is the distribution of $jR_{j,\ell}(f)$ and m_j its expectation.

If the variance is infinite, then again the Gaussian limit law holds:

Theorem 1.4.2. *Let f be of bounded variation so that for any $\theta > 0$*

$$\sum_{j=1}^{\infty} \theta j \mathbb{E}\left[R_{j,1}^2(f)\right] = \infty. \tag{1.4.13}$$

Then,

1. *for σ chosen with respect to the Ewens distribution with paramter θ,*

$$\frac{I_{\sigma,N,z}(f) - \mathbb{E}\left[I_{\sigma,N,z}(f)\right]}{\sqrt{\operatorname{Var} I_{\sigma,N,z}(f)}} \tag{1.4.14}$$

   converges in distribution to $\mathcal{N}(0,1)$.

2. *The asymptotic behavior of the expectation is given by*

$$\mathbb{E}\left[I_{\sigma,N,z}(f)\right] = N \int_0^1 f(x)dx + \sum_{j=1}^{N} \theta \mathbb{E}\left[R_{j,1}(f)\right] + O(1). \tag{1.4.15}$$

3. *The asymptotic behavior of the variance is given by*

$$\operatorname{Var} I_{\sigma,N,z}(f) \sim \sum_{j=1}^{N} \theta j \mathbb{E}\left[R_{j,1}^2(f)\right]. \tag{1.4.16}$$

We will give the proofs of the two theorems in Chapter 4. The results are strongly comparable with the results given in [43]. For the case of the infinite variance, they give the conditions in terms of a p-Norm instead of the sup-Norm given in Chapter 4 (see Theorem 6.2 [43]). Moreover, for the case of finite variance, they also give an infinitely divisible limit law for matrices of the form (1.3.5), where the random permutation has cycle weights (see Theorem 5.5. [43]). Weighted random permutations has been studied for instance by [11], [12] or [34]. This model appears in the study of large systems of quantum bosonic particles [11], where the parameters θ depend on quantities such as the temperature, the density and the particle interactions. For weighted random permutations, the techniques from [10] and [43] do not apply in the case of the infinite variance, where a Gaussian limit is expected.

Chapter 2

Fluctuations of linear statistics

Smooth linear statistics of random permutation matrices, sampled under a general Ewens distribution, exhibit an interesting non-universality phenomenon. Though they have bounded variance, their fluctuations are asymptotically non-Gaussian but infinitely divisible. The fluctuations are asymptotically Gaussian for less smooth linear statistics for which the variance diverges. The degree of smoothness is measured in terms of the quality of the trapezoidal approximations of the integral of the observable.

2.1 Introduction

We study the fluctuations of the spectrum of random permutation matrices, or more precisely, of their linear statistics under the Ewens distribution for a wide class of functions. The study of linear statistics of the spectrum of random matrices is an active field (for results concerning invariant ensembles, see for instance [17], [24], [25], [27], [45], [46], [47], [59] or [77] and for non-invariant ensembles see for instance [6], [8], [21], [16], [28], [51], [54] or [76]). All previous results (except [59]) have two common features. Firstly, the variance of linear statistics does not diverge for smooth enough functions, and thus, no normalization is needed to get a limit law, whereas for less smooth functions the variance blows up very slowly (i.e. logarithmically). The second feature is that those fluctuations are asymptotically Gaussian (except in [59] again, where invariant ensembles with more than one cut are shown to have non-Gaussian fluctuations). We will see that the behavior of the variance of the linear statistics of random permutation matrices follow the general pattern. But we will also prove that the asymptotic limit law is more surprising, in that it is not Gaussian but infinitely divisible when the function is smooth enough. This is in contrast to the case where the function is less regular, the fluctuations being then indeed asymptotically Gaussian. This Gaussian behavior was previously proved by K. Wieand (see [76]) for the special case where the linear statistic is the number of eigenvalues in a given arc, and for uniformly distributed permutations.

We first introduce our notations. If N is an integer, $\mathcal{S}_N$ will denote the symmetric group. We denote by M_σ the permutation matrix defined by the permutation $\sigma \in \mathcal{S}_N$. For any $1 \leq i, j \leq N$, the entry $M_\sigma(i,j)$ is given by

$$M_\sigma(i,j) = \mathbb{1}_{i=\sigma(j)}. \tag{2.1.1}$$

M_σ is unitary, its eigenvalues belong to the unit circle $\mathbb{T}$. We denote them by

$$\lambda_1(\sigma) = e^{2i\pi\varphi_1(\sigma)}, \dots, \lambda_N(\sigma) = e^{2i\pi\varphi_N(\sigma)} \in \mathbb{T}, \tag{2.1.2}$$

where $\varphi_1(\sigma), \dots, \varphi_N(\sigma)$ are in $[0,1]$.
For any real-valued periodic function f of period 1, we define the linear statistic

$$I_{\sigma,N}(f) := \mathrm{Tr}\widetilde{f}(M_\sigma) = \sum_{i=1}^{N} f(\varphi_i(\sigma)), \tag{2.1.3}$$

where $\widetilde{f}(e^{2i\pi\varphi}) = f(\varphi)$ is a function on the unit circle $\mathbb{T}$.
We consider random permutation matrices by sampling σ under the Ewens distribution

$$\nu_{N,\theta}(\sigma) = \frac{\theta^{K(\sigma)}}{\theta(\theta+1)\dots(\theta+N-1)}, \tag{2.1.4}$$

where $\theta > 0$ and $K(\sigma)$ is the total number of cycles of the permutation σ. The case $\theta = 1$ corresponds to the uniform measure on $\mathcal{S}_N$.

We study here the asymptotic behavior of the linear statistic $I_{\sigma,N}(f)$ under the Ewens distribution $\nu_{N,\theta}$ for any $\theta > 0$, and a wide class of functions f. As mentioned above, the asymptotic behavior depends strongly on the smoothness of f. In order to quantify this dependence, we introduce the sequence

$$R_j(f) = \frac{1}{j}\sum_{k=0}^{j-1} f\left(\frac{k}{j}\right) - \int_0^1 f(x)dx. \tag{2.1.5}$$

Using the periodicity of f, it is clear that

$$R_j(f) = \frac{1}{j}\left(\frac{1}{2}f(0) + \sum_{k=1}^{j-1} f\left(\frac{k}{j}\right) + \frac{1}{2}f(1)\right) - \int_0^1 f(x)dx. \tag{2.1.6}$$

So that $R_j(f)$ is easily seen to be the error in the composite trapezoidal approximation to the integral of f [20].
We will see that the asymptotic behavior of the linear statistic $I_{\sigma,N}(f)$ is controlled by the asymptotic behavior of the $R_j(f)$'s, when j tends to infinity, i.e. by the quality of the composite trapezoidal approximation to the integral of f. The role played by the quality of the trapezoidal approximation of f might seem surprising, but it is in fact very natural. It is a simple consequence of the fact that the spectrum of the permutation matrix M_σ is easily expressed in terms of the cycle counts of the random permutation σ, i.e. the numbers $\alpha_j(\sigma)$ of cycles of length j, for $1 \leq j \leq N$. Indeed, the spectrum of M_σ consists in the union, for $1 \leq j \leq N$, of the sets of j-th roots of unity, each taken with multiplicity $\alpha_j(\sigma)$. This gives

$$I_{\sigma,N}(f) = \sum_{j=1}^{N} \alpha_j(\sigma) \sum_{\omega^j=1} \widetilde{f}(\omega) = \sum_{j=1}^{N} \alpha_j(\sigma) \sum_{k=0}^{j-1} f\left(\frac{k}{j}\right). \tag{2.1.7}$$

So that, using the definition (4.1.9) of the R_j's, and the obvious fact that $\sum_{j=1}^{N} j\alpha_j(\sigma) = N$, it becomes clear that:

$$I_{\sigma,N}(f) = N \int_0^1 f(x)dx + \sum_{j=1}^{N} \alpha_j(\sigma) j R_j(f). \tag{2.1.8}$$

At this point, and using the basic equality (2.1.8), it is easy to explain intuitively the non-universality phenomenon we have uncovered in this work. When the function f is smooth enough, the sequence $R_j(f)$ converges fast enough to zero to ensure that the linear statistic is well approximated by the first terms in the sum (2.1.8). These terms correspond to the well separated eigenvalues associated with small cycles. The discrete effects related to these small cycles in the spectrum are then dominant, and are responsible for the non-Gaussian behavior. Thus, the appearance of non-universal fluctuations is due to a very drastic localization phenomenon. Indeed, the important eigenvalues for the behavior of smooth linear statistics are atypical in the sense that they correspond to very localized eigenvectors, those localized on small cycles. When the function is less smooth, the variance will diverge (slowly) so that a normalization will be necessary. After this normalization, the discrete effects will be washed away and the limit law will be Gaussian.

We will first describe the fluctuations of linear statistics of smooth enough functions f, i.e. in the case when the $R_j(f)$'s decay to 0 fast enough to ensure that the variance of the linear statistic stays bounded.

Theorem 2.1.1. *Let θ be any positive number, and f be a function of bounded variation. Assume that*

$$\sum_{j=1}^{\infty} jR_j(f)^2 \in (0, \infty). \tag{2.1.9}$$

Then,

1. *under the Ewens distribution $\nu_{N,\theta}$, the distribution of the centered linear statistic*

$$I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]$$

   converges weakly, as N goes to infinity, to a non-Gaussian infinitely divisible distribution $\mu_{f,\theta}$.

2. *The distribution $\mu_{f,\theta}$ is defined by its Fourier transform*

$$\widehat{\mu}_{f,\theta}(t) = \exp\left(\theta \int (e^{itx} - 1 - itx) dM_f(x)\right), \tag{2.1.10}$$

   where the Lévy measure M_f is given by

$$M_f = \sum_{j=1}^{\infty} \frac{1}{j} \delta_{jR_j(f)}. \tag{2.1.11}$$

3. *The asymptotic behavior of the expectation of the linear statistic is given by*

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N\int_0^1 f(x)dx + \theta\sum_{j=1}^{N} R_j(f) + o(1). \tag{2.1.12}$$

Here, the second term $\sum_{j=1}^{N} R_j(f)$ *may diverge, but not faster than logarithmically.*

$$\sum_{j=1}^{N} R_j(f) = O(\sqrt{\log N}) \tag{2.1.13}$$

4. *The asymptotic behavior of the variance of the linear statistic is given by*

$$\mathrm{Var}[I_{\sigma,N}(f)] = \theta\sum_{j=1}^{N} jR_j(f)^2 + o(1). \tag{2.1.14}$$

Remark 2.1.1.1. *In this theorem (and in the next), we restrict ourselves to the class of functions f of bounded variation. This is not at all a necessary hypothesis, but it simplifies greatly the statements of the theorems. Our proofs give more. We will come back later (in Section 2) to the best possible assumptions really needed for each statement. These assumptions involve the notion of Cesaro means of fractional order, which we wanted to avoid in this introduction.*

Remark 2.1.1.2. *We note that the assumption* (2.1.9) *is not satisfied in the trivial case where f is in the kernel of the composite trapezoidal rule, i.e when the composite trapezoidal rule gives the exact approximation to the integral of f for all j's. In this case, the sequence $R_j(f)$ is identically zero and the linear statistic is non-random. Obviously, this is the case for every constant function f and for every odd function f, i.e. if*

$$f(x) = -f(1-x). \tag{2.1.15}$$

It is indeed easy to see then that $R_j(f) = 0$ for all $j \geq 1$.

Remark 2.1.1.3. *Consider now the even part of f, i.e.*

$$f_{even}(x) = \frac{1}{2}\left(f(x) + f(1-x)\right). \tag{2.1.16}$$

It is clear then that

$$R_j(f) = R_j(f_{even}), \tag{2.1.17}$$

so that the assumption (2.1.9) *in fact only deals with the even part of f.*

Remark 2.1.1.4. *In order to avoid the possibility mentioned above for all $R_j(f)$'s to be zero, we introduce the following assumption*

$$f_{even} \text{ is not a constant.} \tag{2.1.18}$$

Note that, in general, it is not true that (2.1.18) *implies that the sequence of $R_j(f)$'s is not identically zero, even when f is continuous! (See [41] or [53].) But when f is in the Wiener algebra, i.e. when its Fourier series converges absolutely, then* (2.1.18) *does imply that one of the $R_j(f)$'s is non zero (see [53], p. 260).*

Remark 2.1.1.5. *It is in fact easy to compute explicitly the value of the expectation and variance of the linear statistic $I_{\sigma,N}(f)$ for any value of θ and of N. This is done below, in Section 2.7. The asymptotic analysis is not immediate for the values of $\theta < 1$.*

We now want to show how the assumption (2.1.9) can easily be translated purely in terms of manageable regularity assumptions on the function f itself.

Corollary 2.1.1.1. *If $f \in \mathcal{C}^1$, let $\omega(f', \delta)$ be the modulus of continuity of its derivative f'. Assume that*

$$\sum_{j=1}^{\infty} \frac{1}{j} \omega(f', 1/j)^2 < \infty, \tag{2.1.19}$$

also assume (2.1.18) in order to avoid the trivial case mentioned above, then the conclusions of Theorem 2.1.1 hold.

Of course, the condition (2.1.19) is satisfied if $f \in \mathcal{C}^{1+\alpha}$, for $0 < \alpha < 1$, i.e. if f' is α-Hölder continuous.
We can give variants of the assumptions of smoothness of f given in Corollary 2.1.1.1. For instance,

Corollary 2.1.1.2. *If f has a derivative in L^p, let $\omega^{(p)}(f', \delta)$ be the modulus of continuity in L^p of its derivative f', i.e.*

$$\omega^{(p)}(f', \delta) = \sup_{0 \leq h \leq \delta} \left\{ \int_0^1 |f'(x+h) - f'(x)|^p \right\}^{1/p}. \tag{2.1.20}$$

Assume that

$$\omega^{(p)}(f', \delta) \leq \delta^{\alpha} \quad \text{with } \alpha > \frac{1}{p}, \tag{2.1.21}$$

also assume (2.1.18) in order to avoid the trivial case mentioned above, then the conclusions of Theorem 2.1.1 hold.

It is of course also possible to relate the $R_j(f)$'s to the Fourier coefficients of f. Indeed, if the Fourier series of f

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n2\pi x) + \sum_{n=1}^{\infty} b_n \sin(n2\pi x) \tag{2.1.22}$$

converges, then the Poisson summation formula shows that

$$R_j(f) = \sum_{n=1}^{\infty} a_{jn}. \tag{2.1.23}$$

Using this relation, it is easy to prove the following Corollary:

Corollary 2.1.1.3. *If f is in the Sobolev space H^s, for $s > 1$, and if one assumes (2.1.18), then the conclusions of Theorem 2.1.1 hold.*

Remark 2.1.1.6. *The formula 2.1.23 gives an expression for the asymptotic variance of the linear statistic*

$$\lim_{N\to\infty} \mathrm{Var}[I_{\sigma,N}(f)] = \theta\sum_{j=1}^{\infty} jR_j(f)^2 = \theta\sum_{k,l=1}^{\infty} a_k a_l d(k,l), \tag{2.1.24}$$

where $d(k,l)$ is the sum of the divisors of the integers k and l.

We now give two interesting examples of functions satisfying the conditions of Theorem 2.1.1:

Example 2.1.2. Let f be a trigonometric polynomial of degree k. Then, $R_j(f) = 0$ for all $j > k$. Obviously, the condition (2.1.9) of Theorem 2.1.1 is satisfied and the limit distribution $\mu_{f,\theta}$ is a compound Poisson distribution with M_f given by

$$M_f = \theta\sum_{j=1}^{k}\frac{1}{j}\delta_{jR_j}. \tag{2.1.25}$$

Example 2.1.3. Let $f \in \mathcal{C}^\infty$ and $f \equiv 1$ on $[a,b]$ and $f \equiv 0$ on $[a-\epsilon, b+\epsilon]^c$, then the result of Theorem 2.1.1 applies. So, the centered linear statistic $I_{\sigma_N}(f) - \mathbb{E}[I_{\sigma_N}(f)]$ has a finite variance and a non-Gaussian infinitely divisible limit distribution. This is a very different behavior from the case $f = \mathbb{1}_{[a,b]}$ (see below), where the limit is Gaussian.

We now give our second main result, i.e. sufficient conditions ensuring that the variance of the linear statistic $I_{\sigma,N}(f)$ diverges and that the linear statistic converges in distribution to a Gaussian, when centered and normalized.

Theorem 2.1.4. *Let θ be any positive number, and f be a bounded variation function such that*

$$\sum_{j=1}^{\infty} jR_j(f)^2 = \infty. \tag{2.1.26}$$

Then,

1. *under the Ewens distribution $\nu_{N,\theta}$, the distribution of the centered and normalized linear statistic*
$$\frac{I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]}{\sqrt{\mathrm{Var}\, I_{\sigma,N}(f)}} \tag{2.1.27}$$
converges weakly, as N goes to infinity, to the Gaussian standard distribution $\mathcal{N}(0,1)$.

2. *The asymptotic behavior of the expectation of the linear statistic is given by*

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N\int_0^1 f(x)dx + \theta\sum_{j=1}^{N} R_j(f) + O(1). \tag{2.1.28}$$

Here, the second term $\sum_{j=1}^{N} R_j(f)$ may diverge, but not faster than logarithmically.

$$\sum_{j=1}^{N} R_j(f) = O(\log N) \tag{2.1.29}$$

3. *The asymptotic behavior of the variance of the linear statistic is given by*

$$\mathrm{Var}[I_{\sigma,N}(f)] \sim \theta \sum_{j=1}^{N} j R_j(f)^2. \tag{2.1.30}$$

Example 2.1.5. Consider $f = \mathbb{1}_{(a,b)}$ for an interval $(a,b) \subset [0,1]$. $I_{\sigma,N}(f)$ is then simply the number of eigenvalues in the arc $[e^{2i\pi a}, e^{2i\pi b}]$. The function f is obviously of bounded variation. This example has been treated in the simple case where $\theta = 1$ in [76]. We will see here that Theorem 2.1.4 enables us to extend the results of [76] to any value of $\theta > 0$. Indeed, the error in the composite trapezoidal approximation $R_j(f)$ is very easy to compute for an indicator function:

$$R_j(f) = \frac{1}{j}\left(\{ja\} - \{jb\}\right). \tag{2.1.31}$$

Obviously, in this case, Theorem 2.1.4 applies and we have that

$$\frac{I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]}{\sqrt{C\theta \log N}} \overset{(d)}{\Rightarrow} \mathcal{N}(0,1).$$

We can also deduce the asymptotic behavior of the expectation and of the variance, using the conclusions of Theorem 2.1.4 and the computations made in the particular case $\theta = 1$ in [76]. Indeed, it is shown in [76], that for a constant $c_1(a,b)$

$$\sum_{j=1}^{N} R_j(f) = -c_1(a,b) \log N + o(\log N). \tag{2.1.32}$$

So that from the statement proven in Theorem 2.1.4 :

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N \int_0^1 f(x)dx + \theta \sum_{j=1}^{N} R_j(f) + O(1). \tag{2.1.33}$$

We see that

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N(b-a) - \theta c_1 \log N + o(\log N). \tag{2.1.34}$$

The value of $c_1(a,b)$ is studied in [76]. It depends on the fact that a and b are rational or not. It vanishes if a and b are both irrational.
We also have, from the computations in [76], that there exists a positive constant $c_2(a,b)$ such that

$$\sum_{j=1}^{N} j R_j(f)^2 = c_2(a,b) \log N + o(\log N). \tag{2.1.35}$$

So that we have for any $\theta > 0$, by Theorem 2.1.4, that

$$\operatorname{Var} I_{\sigma,N}(f) \sim c_2(a,b)\theta \log N.$$

The value of $c_2(a,b)$ also depends on the arithmetic properties of a and b, and is studied in [76].

Remark 2.1.5.1. *We want to point out that f being of bounded variation is not a necessary condition in order to get a Gaussian limit distribution. But, when f is of bounded variation, it is easy to see that there exists a constant C such that*

$$\operatorname{Var} I_{\sigma,N}(f) \leq C \log N. \tag{2.1.36}$$

The case treated in the example above gives the maximal normalization for functions of bounded variation.

The remainder of this article is organized as follows. In Section 2.2, we state our results with weaker assumptions than the theorems given in this introduction. These assumptions use the classical notion of Cesaro means of fractional order, which we recall in the first subsection of Section 2.2. In Section 2.3, we prove the Corollaries 2.1.1.1, 2.1.1.2 and 2.1.1.3, using estimates on the trapezoidal approximation. In order to prove the main results of Section 2.2, our main tool will be the Feller coupling. This is natural since the problem is translated by the basic equality (2.1.8) in terms of cycle counts of random permutations. In Section 2.4, we will need to improve on the known bounds for the approximation given by this coupling (see for example [2] or [9]) and relate these bounds to Cesaro means. We will then be ready to prove in Section 2.5 and Section 2.6 our general results as stated in Section 2.2 and that these more general results imply the two theorems of this introduction Theorems 2.1.1 and 2.1.4.. Finally in the very short Section 2.7, we give an explicit expression for the expectation and variance of the linear statistics as promised in Remark 2.1.1.5.

2.2 Cesaro means and convergence of linear statistics

2.2.1 Cesaro Means

We will state here our optimal results in terms of convergence of the Cesaro means of fractional order. First, we will need to recall the classical notion of Cesaro means of order θ and of Cesaro convergence (C,θ) for a sequence of real numbers, say $s = (s_j)_{j\geq 0}$ (see [85], Volume 1, p. 77, formulae (1.14) and (1.15)).

Definition 2.2.1. *(i) The Cesaro numbers of order $\alpha > -1$ are given by*

$$A_N^{\alpha} := \binom{N+\alpha}{N}. \tag{2.2.1}$$

(ii) The Cesaro mean of order $\theta > 0$ of the sequence $s = (s_j)_{j\geq 0}$ is given by

$$\sigma_N^\theta(s) = \sum_{j=0}^{N} \frac{A_{N-j}^{\theta-1}}{A_N^\theta} s_j. \tag{2.2.2}$$

(iii) A sequence of real numbers $s = (s_j)_{j\geq 0}$ is said to be convergent in Cesaro sense of order θ (or in (C,θ) sense) to a limit ℓ iff the sequence of Cesaro means $\sigma_N^\theta(s)$ converges to ℓ.

Let us recall the following basic facts about Cesaro convergence (see [85]):

Lemma 2.2.2. *(i) Convergence in the (C,θ_1) sense to a limit ℓ, implies convergence (C,θ_2) to the same limit for any $\theta_1 \leq \theta_2$.*

(ii) Usual convergence is (C,0) convergence. The classical Cesaro convergence is (C,1) convergence.

(iii) If the sequence $(s_j)_{j\geq 0}$ is bounded and converges (C,θ_1) to a limit ℓ for some value $\theta_1 > 0$, then it converges (C,θ) to the same limit, for any $\theta > 0$.

These facts are all classical, see [85] for a proof, in particular Lemma (2.27), p. 70, Volume 2 for a proof of (iii).

2.2.2 The case of bounded variance, non-Gaussian limits

We will give here a sharper statement than Theorem 2.1.1 and prove that it implies Theorem 2.1.1. Define the sequence $u(f) = (u_j(f))_{j\geq 1} = (jR_j(f))_{j\geq 1}$.

Theorem 2.2.3. *Let θ be any positive number, and assume that the sequence $|u(f)| = (|u_j(f)|)_{j\geq 1}$ converges to zero in the Cesaro (C,θ) sense if $\theta < 1$. Also assume that*

$$\sum_{j=1}^{\infty} jR_j(f)^2 \in (0,\infty). \tag{2.2.3}$$

Then,

1. *under the Ewens distribution $\nu_{N,\theta}$, the distribution of the centered linear statistic*

$$I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]$$

converges weakly, as N goes to infinity, to a non-Gaussian infinitely divisible distribution $\mu_{f,\theta}$.

2. *The distribution $\mu_{f,\theta}$ is defined by its Fourier transform*

$$\widehat{\mu}_{f,\theta}(t) = \exp\left(\theta \int (e^{itx} - 1 - itx) dM_f(x)\right), \tag{2.2.4}$$

where the Lévy measure M_f is given by

$$M_f = \sum_{j=1}^{\infty} \frac{1}{j} \delta_{jR_j(f)}. \tag{2.2.5}$$

3. *The asymptotic behavior of the expectation of the linear statistic is given by*

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N \int_0^1 f(x)dx + \sum_{j=1}^{N} R_j(f) + o(1). \tag{2.2.6}$$

Here, the second term $\sum_{j=1}^{N} R_j(f)$ may diverge, but not faster than logarithmically.

$$\sum_{j=1}^{N} R_j(f) = O(\sqrt{\log N}) \tag{2.2.7}$$

4. *If, on top of the preceding assumptions, one assumes that the sequence $u(f)^2 = (u_j(f)^2)_{j\geq 1}$ converges in Cesaro $(C, 1\wedge\theta)$ sense, then the asymptotic behavior of the variance of the linear statistic is given by*

$$\mathrm{Var}[I_{\sigma,N}(f)] = \theta \sum_{j=1}^{N} jR_j(f)^2 + o(1). \tag{2.2.8}$$

Theorem 2.2.3 will be proved in Section 5.

2.2.3 The case of unbounded variance, Gaussian limits

We will give here a slightly sharper statement than Theorem 2.1.4 and prove that it implies Theorem 2.1.4.

Theorem 2.2.4. *Let θ be any positive number, and assume that*

$$\sum_{j=1}^{\infty} jR_j(f)^2 = \infty \tag{2.2.9}$$

and that

$$\max_{1\leq j\leq N} |jR_j| = o(\eta_N), \tag{2.2.10}$$

where $\eta_N^2 = \theta \sum_{j=1}^{N} jR_j(f)^2$. Then,

1. *under the Ewens distribution $\nu_{N,\theta}$, the distribution of the centered and normalized linear statistic*

$$\frac{I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]}{\sqrt{\mathrm{Var}\, I_{\sigma,N}(f)}} \tag{2.2.11}$$

converges weakly, as N goes to infinity, to the Gaussian standard distribution $\mathcal{N}(0,1)$.

2. *The asymptotic behavior of the expectation of the linear statistic is given by*

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N\int_0^1 f(x)dx + \sum_{j=1}^{N} R_j(f) + o(\eta_N). \tag{2.2.12}$$

Here, the second term $\sum_{j=1}^{N} R_j(f)$ may diverge, but not faster than logarithmically.

$$\sum_{j=1}^{N} R_j(f) = o(\eta_N\sqrt{\log N}) \tag{2.2.13}$$

3. *The asymptotic behavior of the variance of the linear statistic is given by*

$$\mathrm{Var}[I_{\sigma,N}(f)] \sim \eta_N^2 = \theta\sum_{j=1}^{N} jR_j(f)^2. \tag{2.2.14}$$

This theorem will be proved in Section 6.

2.3 Estimates on the trapezoidal rule and proofs of the Corollaries 2.1.1.1, 2.1.1.2 and 2.1.1.3

In this section, we will discuss known results about the quality of the composite trapezoidal approximation for periodic functions, in order to relate the decay of the $R_j(f)$'s to the regularity of f. Moreover, we will give proofs of Corollary 2.1.1.1, Corollary 2.1.1.2, Corollary 2.1.1.3.

2.3.1 Jackson-type estimates on the composite trapezoidal approximation

In order to relate the decay of the $R_j(f)$'s to the regularity of f, we can use two related approaches. First, we can control directly the size of the R_j's by Jackson type inequalities as in [15], [18] or [62]. Or we may use the Poisson summation formula given in (2.1.23) and use the decay of the Fourier coefficients of f.

We start by using the first approach, and recall known Jackson-type estimates of the error in the trapezoidal approximation.

Lemma 2.3.1. *(i) There exists a constant $C \leq 179/180$ such that*

$$|R_j(f)| \leq C\omega_2(f, 1/(2j)), \tag{2.3.1}$$

where

$$\omega_2(f,\delta) = \sup_{|h|\leq\delta, x\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|. \tag{2.3.2}$$

(ii) If the function f is in C^1, then

$$|R_j(f)| \leq C\frac{\omega(f', 1/j)}{2j}. \tag{2.3.3}$$

(iii) If the function f is in $W^{1,p}$, then

$$|R_j(f)| \leq C\omega^{(p)}(f', 1/j)\frac{1}{j^{1-1/p}}. \tag{2.3.4}$$

Proof. The first item is well known, see (see [15]).
The second item is a consequence of the first, since by the Mean Value Theorem

$$\omega_2(f, \delta) \leq \delta\omega(f', 2\delta). \tag{2.3.5}$$

The third item is also an easy consequence of the first since

$$f(x+2h) - 2f(x+h) + f(x) = \int_x^{x+h} (f'(t+h) - f'(t))dt. \tag{2.3.6}$$

So that

$$|f(x+2h) - 2f(x+h) + f(x)| \leq \left(\int_0^1 |f'(t+h) - f'(t)|^p dt\right)^{\frac{1}{p}} h^{\frac{p-1}{p}}, \tag{2.3.7}$$

which shows that

$$\omega_2(f, \delta) \leq \omega^{(p)}(f', \delta)\delta^{1-1/p}. \tag{2.3.8}$$

□

2.3.2 Proofs of Corollary 2.1.1.1 and 2.1.1.2 using Jackson bounds

Proof of Corollary 2.1.1.1. We can control the decay of the $R_j(f)$'s using the item (ii) of Lemma 2.3.1, which implies that

$$jR_j(f)^2 \leq C^2\frac{\omega(f', 1/j)^2}{4j} \tag{2.3.9}$$

It is then clear that under the assumption (2.1.19), the series $\sum_{j=1}^{\infty} jR_j(f)^2$ is convergent. But (2.1.19) implies that the Fourier series of f is absolutely convergent, so by the result mentioned above ([53], p. 260) it is true that (2.1.18) implies that one of the $R_j(f)$'s is non zero. And thus, $\sum_{j=1}^{\infty} jR_j(f)^2 \in (0, \infty)$. If we add that f is obviously of bounded variation, we have then checked the assumptions of Theorem 2.1.1 and thus, proved Corollary 2.1.1.1. □

Proof of Corollary 2.1.1.2. We can here control the decay of the $R_j(f)$'s using the item (iii) of Lemma 2.3.1, and the assumption (2.1.21), which imply that

$$|R_j(f)| \leq C\frac{1}{j^{1+\alpha-1/p}}. \tag{2.3.10}$$

So, if $\alpha > 1/p$, the series $\sum_{j=1}^{\infty} jR_j(f)^2$ is convergent, since

$$jR_j^2(f) \leq \frac{C}{j^{1+2(\alpha-1/p)}}. \tag{2.3.11}$$

Moreover, as above, it is easy to see that (2.1.21) implies that the Fourier series of f is absolutely convergent, so by the result mentioned above ([53], p. 260) it is true that (2.1.18) implies that one of the $R_j(f)$'s is non zero. Again, f is obviously of bounded variation, we have then checked the assumptions of Theorem 2.1.1 and thus, proved Corollary 2.1.1.2. □

Remark: It is in fact true that $\lim_{j\to\infty} jR_j(f) = 0$ is satisfied as soon as $f \in W^{1,p}$ (see [18]).

2.3.3 Proof of Corollary 2.1.1.3 using the Poisson summation formula

We now turn to the proof of Corollary 2.1.1.3, using the second possible approach, i.e. the Poisson Summation Formula, (2.1.23).

Proof of Corollary 2.1.1.3. Let f be in H^s, $s > 1$ and consider its Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n2\pi x) + \sum_{n=1}^{\infty} b_n \sin(n2\pi x). \tag{2.3.12}$$

Then there exists a sequence $(c_k)_{k\geq 1} \in \ell^2$ such that

$$a_k = \frac{c_k}{k^s}. \tag{2.3.13}$$

So,

$$C_j := \sum_{\ell\geq 1} \frac{c_{j\ell}}{\ell^s} \tag{2.3.14}$$

is in ℓ^2 by Lemma 4 of [62], p. 131. Thus, using the Poisson summation formula (2.1.23),

$$R_j(f) = \frac{C_j}{j^s}, \tag{2.3.15}$$

which is more than enough to prove that the series $\sum_{j=1}^{\infty} jR_j(f)^2$ is convergent. Moreover, as above, it is easy to see that (2.1.18) implies that one of the $R_j(f)$'s is non zero, and that f is obviously of bounded variation. We have then checked the assumptions of Theorem 2.1.1 and thus, proved Corollary 2.1.1.3. □

2.4 Bounds on the Feller coupling and Cesaro Means

2.4.1 The Feller Coupling

Let $\sigma \in \mathcal{S}_N$ be a given permutation and $\alpha_j(\sigma)$ be the number of j-cycles of σ. A classical result is that under the Ewens distribution $\nu_{N,\theta}$, the joint distribution of $(\alpha_1(\sigma), \dots, \alpha_N(\sigma))$ is given by

$$\nu_{N,\theta}[(\alpha_1(\sigma), \dots, \alpha_N(\sigma)) = (a_1, \dots, a_N)] = \mathbb{1}_{\sum_{j=1}^N ja_j = N} \frac{N!}{\theta_{(N)}} \prod_{j=1}^N \left(\frac{\theta}{j}\right)^{a_j} \frac{1}{a_j!}, \tag{2.4.1}$$

where $\theta_{(N)} = \theta(\theta+1)\dots(\theta+N-1)$.

We recall now the definition and some properties of the Feller coupling, a very useful tool to study the asymptotic behavior of $\alpha_j(\sigma)$ (see for example [2], p. 523).
Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $(\xi_i)_{i\geq 1}$ of independent Bernoulli random variables defined on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}[\xi_i = 1] = \frac{\theta}{\theta + i - 1} \quad \text{and} \quad \mathbb{P}[\xi_i = 0] = \frac{i-1}{\theta + i - 1}.$$

For $1 \leq j \leq N$, denote the number of spacings of length j in the sequence $1\xi_2 \cdots \xi_N 1$ by $C_j(N)$, i.e.

$$C_j(N) = \sum_{i=1}^{N-j} \xi_i(1-\xi_{i+1})\dots(1-\xi_{i+j-1})\xi_{i+j} + \xi_{N-j+1}(1-\xi_{N-j+2})\dots(1-\xi_N). \tag{2.4.2}$$

Define $(W_{jm})_{j\geq 1}$ by

$$W_{jm} = \sum_{i=m+1}^{\infty} \xi_i(1-\xi_{i+1})\dots(1-\xi_{i+j-1})\xi_{i+j} \tag{2.4.3}$$

and set for $j \geq 1$,

$$W_j := W_{j0}. \tag{2.4.4}$$

Define

$$J_N = \min\{j \geq 1 : \xi_{N-j+1} = 1\} \tag{2.4.5}$$

and

$$K_N = \min\{j \geq 1 : \xi_{N+j} = 1\}. \tag{2.4.6}$$

With the notations above, we state the following result of [9], p.169:

Theorem 2.4.1. *Under the Ewens distribution $\nu_{N,\theta}$,*

(i) $(C_j(N))_{1\leq j\leq N}$ *has the same distribution as* $(\alpha_j(\sigma))_{1\leq j\leq N}$ *, i.e. for any* $a = (a_1,\dots,a_N)\in\mathbb{N}^N$,

$$\mathbb{P}[(C_1(N),\dots,C_N(N) = a] = \nu_{N,\theta}[(\alpha_1(\sigma),\dots,\alpha_N(\sigma) = a], \tag{2.4.7}$$

(ii) $(W_j)_{1\leq j\leq N}$ *are independent Poisson random variables with mean* θ/j,

(iii) and

$$|C_j(N) - W_j| \leq W_{jN} + \mathbb{1}_{\{J_N+K_N=j+1\}} + \mathbb{1}_{\{J_N=j\}}. \tag{2.4.8}$$

We will need to improve on the known results for the Feller coupling. In particular we will need the following. For any sequence of real numbers $(u_j)_{j\geq 1}$, define

$$G_N = \sum_{j=1}^{N} u_j C_j(N) \tag{2.4.9}$$

and

$$H_N = \sum_{j=1}^{N} u_j W_j. \tag{2.4.10}$$

We will need to control the L^1 and L^2-distances between the random variables G_N and H_N. In order to prove Theorem 2.1.1 and Theorem 2.1.4, we will apply these estimates to the case where the sequence u_j is chosen to be $u_j(f) = jR_j(f)$.

2.4.2 L^1 bounds on the Feller Coupling

We begin with the control of the L^1-distance in this subsection. We first state our result in a very simple (but not optimal) shape.

Lemma 2.4.2. *For every $\theta > 0$, there exists a constant $C(\theta)$ such that, for every integer N,*

$$\mathbb{E}(|G_N - H_N|) \leq C(\theta) \max_{1\leq j\leq N} |u_j|. \tag{2.4.11}$$

This result is a trivial consequence of a deeper result, that we now give after introducing some needed notations. We recall that for any real number x and integer k,

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!} \quad . \tag{2.4.12}$$

We now define for any $\theta > 0$ and every $1\leq j\leq N$,

$$\Psi_N(j) := \binom{N-j+\gamma}{N-j}\binom{N+\gamma}{N}^{-1} = \prod_{k=0}^{j-1}\frac{N-k}{\theta+N-k-1}, \tag{2.4.13}$$

where $\gamma = \theta - 1$.

We then have:

Lemma 2.4.3.

$$\mathbb{E}|G_N - H_N| \le \frac{C(\theta)}{N}\sum_{j=1}^{N}|u_j| + \frac{\theta}{N}\sum_{j=1}^{N}|u_j|\Psi_N(j) \tag{2.4.14}$$

Lemma 2.4.3 is obviously a direct consequence of the following:

Lemma 2.4.4. *Let $\theta > 0$, there exists a constant $C(\theta)$, such that, for every $1 \le j \le N$*

$$\mathbb{E}|C_j(N) - W_j| \le \frac{C(\theta)}{N} + \frac{\theta}{N}\Psi_N(j) \tag{2.4.15}$$

In order to prove Lemma 2.4.4, we note that, by 2.4.8,

$$\mathbb{E}|C_j(N) - W_j| \le \mathbb{E}(W_{jN}) + \mathbb{P}(J_N + K_N = j+1) + \mathbb{P}(J_N = j). \tag{2.4.16}$$

It thus suffices to provide bounds on $\mathbb{E}[W_{jN}]$, $\mathbb{P}[J_N = j]$ and $\mathbb{P}[J_N + K_N = j+1]$.

Lemma 2.4.5. *For any $\theta > 0$ and for every $1 \le j \le N$,*

$$\mathbb{E}(W_{jN}) \le \frac{\theta^2}{N-1}. \tag{2.4.17}$$

Proof. Let

$$U_i^{(j)} := \xi_i(1-\xi_{i+1})\dots(1-\xi_{i+j-1})\xi_{i+j}, \tag{2.4.18}$$

then, for $i \ge 2$,

$$\mathbb{E}(U_i^{(j)}) \le \mathbb{E}(\xi_i)\mathbb{E}(\xi_{i+j}) = \frac{\theta^2}{(\theta+i-1)(\theta+i+j-1)} \le \frac{\theta^2}{(i-1)^2}. \tag{2.4.19}$$

By (2.4.19), we have immediately that, for any $\theta > 0$,

$$\mathbb{E}(W_{jN}) = \sum_{i=N+1}^{\infty} U_i^{(j)} \le \theta^2 \sum_{\ell=N}^{\infty}\frac{1}{\ell^2} \le \frac{\theta^2}{N-1}. \tag{2.4.20}$$

□

We compute next the distribution of the random variable J_N explicitly.

Lemma 2.4.6.

$$\mathbb{P}[J_N = j] = \frac{\theta}{N}\Psi_N(j). \tag{2.4.21}$$

Proof. The random variable J_N is equal to j if and only if $\xi_N = 0, \xi_{N-1} = 0, \dots, \xi_{N-j+2} = 0$ and $\xi_{N-j+1} = 1$. So, for any $1 \le j \le N$,

$$\begin{aligned}\mathbb{P}[J_N = j] &= \frac{N-1}{\theta+N-1} \times \frac{N-2}{\theta+N-2}\dots\frac{N-(j-1)}{\theta+N-(j-1)} \times \frac{\theta}{\theta+N-j} \\ &= \frac{\theta}{N}\prod_{k=0}^{j-1}\frac{N-k}{\theta+N-k-1} = \frac{\theta}{N}\Psi_N(j),\end{aligned} \tag{2.4.22}$$

which proves the claim. □

We now bound the distribution of the random variable $J_N + K_N$.

Lemma 2.4.7. *For any $\theta > 0$,*

$$\mathbb{P}[K_N + J_N = j+1] \leq \frac{\theta}{N}. \tag{2.4.23}$$

Proof. Consider first the random variable K_N. For any $\theta > 0$,

$$\begin{aligned}
\mathbb{P}[K_N = j] &= \frac{N}{\theta+N} \times \frac{N+1}{\theta+N+1} \cdots \frac{N+j-2}{\theta+N+j-2} \times \frac{\theta}{\theta+N+j-1} \\
&\leq \frac{N\theta}{(\theta+N+j-2)(\theta+N+j-1)} \\
&\leq \frac{\theta}{\theta+N+j-1} \leq \frac{\theta}{N+\theta}. \qquad (2.4.24)
\end{aligned}$$

For any $\theta > 0$, use (2.4.24) to write

$$\begin{aligned}
\mathbb{P}[K_N + J_N = j+1] &= \sum_{\ell=1}^{j} \mathbb{P}(J_N = j+1-\ell)\mathbb{P}(K_N = \ell) \\
&\leq \frac{\theta}{N+\theta} \sum_{\ell=1}^{j} \mathbb{P}(J_N = j+1-\ell) \\
&= \frac{\theta}{N+\theta} \mathbb{P}(J_N \leq j) \leq \frac{\theta}{N}. \qquad (2.4.25)
\end{aligned}$$

□

The last three lemmas imply the result of Lemma 2.4.4. We have now controlled the L^1-distance between G_N and H_N.

2.4.3 L^2 bounds on the Feller coupling

We now turn to the control of the L^2-distance between the random variables G_N and H_N. We first state our result in a simple (but not optimal) shape.

Lemma 2.4.8. *For every $\theta > 0$, there exists a constant $C(\theta)$ such that, for every integer N,*

$$\mathbb{E}\left((G_N - H_N)^2\right) \leq C(\theta) \max_{1 \leq j \leq N} |u_j|^2 \tag{2.4.26}$$

This result is an immediate consequence of the following much more precise statement.

Lemma 2.4.9. *For every $\theta > 0$, there exists a constant $C(\theta)$ such that, for every integer N,*

$$\begin{aligned}\mathbb{E}\left((G_N - H_N)^2\right) &\leq C(\theta)\left[(\frac{1}{N}\sum_{j=1}^N |u_j|)^2 + \frac{1}{N}\sum_{j=1}^N |u_j|^2\right. \\ &+ \frac{1}{N^2}\sum_{j=1}^N |u_j| \sum_{k=1}^N |u_k|\Psi_N(k) \\ &+ \left.\frac{1}{N}\sum_{j=1}^N |u_j|^2 \Psi_N(j)\right] \qquad (2.4.27)\end{aligned}$$

Proof. We note that

$$\mathbb{E}\left((G_N - H_N)^2\right) \leq \sum_{j,k=1}^N |u_j||u_k| \mathbb{E}(|C_j - W_j||C_k - W_k|). \qquad (2.4.28)$$

By (2.4.8), for any fixed $1 \leq j, k \leq N$,

$$\begin{aligned}|C_j - W_j||C_k - W_k| \leq\ & W_{j,N}W_{k,N} + W_{j,N}\mathbb{1}_{J_N=k} + W_{j,N}\mathbb{1}_{J_N+K_N=k+1} \\ &+\mathbb{1}_{J_N+K_N=j+1}W_{k,N} + \mathbb{1}_{J_N+K_N=j+1}\mathbb{1}_{J_N=k} \\ &+\mathbb{1}_{J_N+K_N=j+1}\mathbb{1}_{J_N+K_N=k+1} \\ &+\mathbb{1}_{J_N=j}W_{k,N} + \mathbb{1}_{J_N=j}\mathbb{1}_{J_N=k} \\ &+\mathbb{1}_{J_N=j}\mathbb{1}_{J_N+K_N=k+1}. \qquad (2.4.29)\end{aligned}$$

To control (2.4.29), we will give upper bounds for all the terms on the RHS. We start by giving a bound for $\mathbb{E}(W_{j,N}W_{k,N})$: By (2.4.18), we have

$$\begin{aligned}\mathbb{E}(W_{j,N}W_{k,N}) &= \sum_{i,\ell \geq N+1} U_i^{(j)}U_\ell^{(k)} \\ &= \sum_{\substack{i,\ell \geq N+1 \\ i<\ell}} U_i^{(j)}U_\ell^{(k)} + \sum_{\substack{i,\ell \geq N+1 \\ i>\ell}} U_i^{(j)}U_\ell^{(k)} + \sum_{i \geq N+1} U_i^{(j)}U_i^{(k)}.\end{aligned} \qquad (2.4.30)$$

We write the first term on the RHS as follows:

$$\sum_{\substack{i,\ell \geq N+1 \\ i<\ell<i+j}} U_i^{(j)}U_\ell^{(k)} + \sum_{\substack{i,\ell \geq N+1 \\ i+j<\ell}} U_i^{(j)}U_\ell^{(k)} + \sum_{i \geq N+1} U_i^{(j)}U_{i+j}^{(k)}. \qquad (2.4.31)$$

It is easy to see that for any $\ell \in (i, i+j)$, $U_i^{(j)}U_\ell^{(k)} = 0$. If ℓ is strictly larger than $i+j$, then $U_i^{(j)}$ and $U_\ell^{(k)}$ are independent. This gives, using (2.4.19),

$$
\begin{aligned}
\sum_{\substack{i,\ell\geq N+1\\ i+j<\ell}} \mathbb{E}\left(U_i^{(j)}U_\ell^{(k)}\right) &= \sum_{\substack{i,\ell\geq N+1\\ i+j<\ell}} \mathbb{E}\left(U_i^{(j)}\right)\mathbb{E}\left(U_\ell^{(k)}\right)\\
&\leq \sum_{i=N+1}^{\infty} \mathbb{E}\left(U_i^{(j)}\right)\sum_{\ell>i+j}\frac{\theta^2}{(\ell-1)^2}\\
&\leq \sum_{i=N+1}^{\infty}\frac{\theta^4}{(i-1)^2}\frac{1}{(i+j-1)} \leq \frac{C(\theta)}{N^2}. \qquad (2.4.32)
\end{aligned}
$$

Also, by the same argument,

$$
\begin{aligned}
\sum_{i=N+1}^{\infty} U_i^{(j)}U_{i+j}^{(k)} &\leq \sum_{i=N+1}^{\infty}\mathbb{E}(\xi_i)\mathbb{E}(\xi_{i+j})\mathbb{E}(\xi_{i+j+k})\\
&\leq \sum_{i=N+1}^{\infty}\frac{\theta^3}{(i-1)(i+j-1)(i+j+k-1)} \leq \frac{C(\theta)}{N^2}. \qquad (2.4.33)
\end{aligned}
$$

For the bound of (2.4.30), we consider now the second and the third term on the RHS. But the second term can be bounded similarly to the first term. For the third term in (2.4.30), we observe that $U_i^{(j)}U_i^{(k)} = 0$ if $j \neq k$. So, by (2.4.19)

$$
\begin{aligned}
\sum_{i\geq N+1}\mathbb{E}\left(U_i^{(j)}U_i^{(k)}\right) &= \sum_{i\geq N+1}\mathbb{E}\left(\left(U_i^{(j)}\right)^2\right) = \sum_{i\geq N+1}\mathbb{E}\left(U_i^{(j)}\right)\\
&\leq \sum_{i\geq N+1}\frac{\theta}{(i-1)^2} \leq \frac{C(\theta)}{N}. \qquad (2.4.34)
\end{aligned}
$$

This gives

$$
\mathbb{E}(W_{j,N}W_{k,N}) = \begin{cases} C(\theta)/N^2 & \text{if } j\neq k\\ C(\theta)/N & \text{if } j=k. \end{cases} \qquad (2.4.35)
$$

So,

$$
\sum_{j,k=1}^{N}|u_j||u_k|\mathbb{E}(W_{j,N}W_{k,N}) \leq C_1(\theta)\left(\frac{1}{N}\sum_{j=1}^{N}|u_j|\right)^2 + C_2(\theta)\frac{1}{N}\sum_{j=1}^{N}|u_j|^2 \qquad (2.4.36)
$$

Obviously, $W_{j,N}$ and $\mathbb{1}_{J_N=k}$ are independent. So, the expectation of the second term on the RHS in (2.4.29) is bounded as follows:

$$
\mathbb{E}(W_{j,N}\mathbb{1}_{J_N=k}) \leq \frac{C(\theta)}{N}\mathbb{P}(J_N=k). \qquad (2.4.37)
$$

Of course, this bound is also valid for $\mathbb{E}(\mathbb{1}_{J_N=j}W_{k,N})$.

Then,

$$\sum_{j,k=1}^{N} |u_j||u_k|\mathbb{E}(W_{j,N}\mathbb{1}_{J_N=k}) \quad \leq \quad \frac{C(\theta)}{N}\sum_{j=1}^{N}|u_j| \cdot \sum_{k=1}^{N}|u_k|\mathbb{P}.(J_N = k) \tag{2.4.38}$$

For $W_{j,N}\mathbb{1}_{J_N+K_N=k+1}$, we write

$$\begin{aligned}\mathbb{E}(W_{j,N}\mathbb{1}_{J_N+K_N=k+1}) &= \mathbb{E}\left(\sum_{\ell=1}^{k} W_{j,N}\mathbb{1}_{J_N+K_N=k+1}\mathbb{1}_{J_N=\ell}\right)\\ &= \sum_{\ell=1}^{k}\mathbb{E}(W_{j,N+k-\ell}\mathbb{1}_{K_N=k+1-\ell})\mathbb{P}(J_N=\ell).\end{aligned} \tag{2.4.39}$$

But,

$$\begin{aligned}&\mathbb{E}(W_{j,N+k-\ell}\mathbb{1}_{K_N=k+1-\ell})\\ &\quad = \mathbb{E}\left(\sum_{i>N+k+1-\ell} U_i^{(j)}\mathbb{1}_{K_N=k+1-\ell}\right) + \mathbb{E}\left(U_{N+k+1-\ell}^{(j)}\mathbb{1}_{K_N=k+1-\ell}\right),\end{aligned} \tag{2.4.40}$$

where $\sum_{i>N+k+1-\ell} U_i^{(j)}$ and $\mathbb{1}_{K_N=k+1-\ell}$ are independent and

$$\begin{aligned}&\mathbb{E}\left(U_{N+k+1-\ell}^{(j)}\mathbb{1}_{K_N=k+1-\ell}\right)\\ &\quad = \mathbb{E}\left(\xi_{N+k+1-\ell}(1-\xi_{N+k+2-\ell})\dots(1-\xi_{N+k+j-\ell})\xi_{N+k+j+1-\ell}\xi_{N+k+1-\ell}\right)\\ &\quad = \mathbb{E}\left(U_{N+k+1-\ell}^{(j)}\right) \leq \frac{C(\theta)}{(N+k-\ell)^2} \leq \frac{C(\theta)}{N^2}.\end{aligned} \tag{2.4.41}$$

So, by (2.4.24)

$$\mathbb{E}(W_{j,N+k-\ell}\mathbb{1}_{K_N=k+1-\ell}) \quad \leq \quad \frac{C_1(\theta)}{N}\mathbb{P}(K_N = k+1-\ell) + \frac{C_2(\theta)}{N^2} \leq \frac{C(\theta)}{N^2}, \tag{2.4.42}$$

which gives

$$\mathbb{E}(W_{j,N+k-\ell}\mathbb{1}_{K_N=k+1-\ell}) \leq \frac{C(\theta)}{N^2}. \tag{2.4.43}$$

This gives also the bound for $\mathbb{E}(\mathbb{1}_{J_N+K_N=j+1}W_{k,N})$.

Then,

$$\sum_{j,k=1}^{N} |u_j||u_k|\mathbb{E}(W_{j,N}\mathbb{1}_{J_N+K_N=k+1}) \quad \leq \quad C(\theta)\left(\frac{1}{N}\sum_{j=1}^{N}|u_j|\right)^2. \tag{2.4.44}$$

For the remaining terms in (2.4.29), we observe that

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{J_N+K_N=j+1}\mathbb{1}_{J_N=k}) &= \mathbb{E}(\mathbb{1}_{K_N=j+1-k}\mathbb{1}_{J_N=k}) = \mathbb{P}(K_N=j+1-k)\mathbb{P}(J_N=k)\\ &\leq \frac{C(\theta)}{N}\mathbb{P}(J_N=k).\end{aligned} \tag{2.4.45}$$

This applies for $\mathbb{1}_{J_N=j}\mathbb{1}_{J_N+K_N=k+1}$, as well.

Then

$$\sum_{j,k=1}^{N}|u_j||u_k|\mathbb{E}(\mathbb{1}_{J_N+K_N=j+1}\mathbb{1}_{J_N=k}) \quad \leq \quad \frac{C(\theta)}{N}\sum_{j=1}^{N}|u_j|\cdot\sum_{k=1}^{N}|u_k|\mathbb{P}(J_N=k) \tag{2.4.46}$$

Also,

$$\mathbb{1}_{J_N+K_N=j+1}\mathbb{1}_{J_N+K_N=k+1}=\begin{cases}1 & \text{if } k=j\\ 0 & \text{if } k\neq j,\end{cases} \tag{2.4.47}$$

so,

$$\mathbb{E}(\mathbb{1}_{J_N+K_N=j+1}\mathbb{1}_{J_N+K_N=k+1})=\begin{cases}\mathbb{P}(J_N+K_N=j+1) & \text{if } k=j\\ 0 & \text{if } k\neq j\end{cases} \tag{2.4.48}$$

and by (2.4.23)

$$\begin{aligned}\sum_{j,k=1}^{N}|u_j||u_k|\mathbb{E}(\mathbb{1}_{J_N+K_N=j+1}\mathbb{1}_{J_N+K_N=k+1}) \quad &= \quad \sum_{j=1}^{N}|u_j|^2\mathbb{P}(J_N+K_N=j+1)\\ &\leq \quad \frac{C(\theta)}{N}\sum_{j=1}^{N}|u_j|^2.\end{aligned} \tag{2.4.49}$$

It is obvious that $\mathbb{1}_{J_N=j}\mathbb{1}_{J_N=k}=0$ for $k\neq j$. So,

$$\mathbb{E}(\mathbb{1}_{J_N=j}\mathbb{1}_{J_N=k})=\begin{cases}\mathbb{P}(J_N=j) & \text{if } k=j\\ 0 & \text{otherwise.}\end{cases} \tag{2.4.50}$$

Then,

$$\sum_{j,k=1}^{N}|u_j||u_k|\mathbb{E}(\mathbb{1}_{J_N=j}\mathbb{1}_{J_N=k})\leq\sum_{j=1}^{N}|u_j|^2\mathbb{P}(J_N=j), \tag{2.4.51}$$

which, using also Lemma 2.4.6, proves the claim of Lemma 2.4.9. □

2.4.4 Cesaro Means and the Feller Coupling Bounds

The link between our estimates and Cesaro means of fractional order is given by an interesting interpretation of Cesaro means of order θ in terms of the random variable J_N.

Lemma 2.4.10. *The Cesaro mean σ_N^θ of order θ of a sequence $s=(s_j)_{j\geq 0}$, with $s_0=0$, is given by*

$$\sigma_N^\theta(s)=\frac{N}{N+\theta}\sum_{j=1}^{N}s_j\mathbb{P}[J_N=j]=\frac{\theta}{N+\theta}\sum_{j=1}^{N}s_j\Psi_N(j) \tag{2.4.52}$$

The proof of this lemma is immediate from Lemma 2.4.6, the Definition 2.2.1 of the Cesaro means and of the numbers $\Psi_N(j)$, given in (2.4.13).
Using this interpretation of the Cesaro means, we can state our results about the L^1 and L^2 distance between the variables G_N and H_N given in Lemma 2.4.3 and Lemma 2.4.9 in terms of the Cesaro means of the sequence $u_j(f)$ and $u_j(f)^2$.

Theorem 2.4.11. *For an $\theta > 0$, there exists a constant $C(\theta)$ such that*

(i)

$$\mathbb{E}|G_N - H_N| \leq C(\theta)(\sigma_N^1(|u|) + \sigma_N^\theta(|u|)) \tag{2.4.53}$$

(ii) and

$$\mathbb{E}\left((G_N - H_N)^2\right) \leq C(\theta)[\sigma_N^1(|u|)^2 + \sigma_N^1(u^2) + \sigma_N^1(|u|)\sigma_N^\theta(|u|) + \sigma_N^\theta(u^2)]. \tag{2.4.54}$$

This theorem is simply a rewriting of Lemma 2.4.3, and Lemma 2.4.9, using the identification given in Lemma 2.4.10. It implies easily the following results

Theorem 2.4.12. *If the sequence $(u_j)_{j\geq 1}$ converges in Cesaro $(C, \theta \wedge 1)$ sense to 0, then*

$$\lim_{N\to\infty} \mathbb{E}|G_N - H_N| = 0. \tag{2.4.55}$$

Proof. By assumption, the sequence converges in $(C, 1)$ and in (C, θ) sense to 0. Thus, the RHS of the bound given in Theorem 2.4.11 tends to zero, which proves Theorem 2.4.12.

□

Similarly we can get the following result about convergence in L^2.

Theorem 2.4.13. *If the sequences $(|u_j|)_{j\geq 1}$ and $(u_j^2)_{j\geq 1}$ both converge to zero in Cesaro $(C, \theta \wedge 1)$, then*

$$\lim_{N\to\infty} \mathbb{E}((G_N - H_N)^2) = 0 \tag{2.4.56}$$

Proof. By assumption, the sequences $(|u_j|)_{j\geq 1}$ and $(u_j^2)_{j\geq 1}$ converge in $(C, 1)$ and in (C, θ) sense to 0. Thus, the RHS of the bound given in (4.2.19) tends to zero, which proves Theorem 2.4.13. □

2.5 Proofs of Theorem 2.2.3 and Theorem 2.1.1

2.5.1 A simple convergence result for series of Poisson random variables

We give here a result of convergence in distribution for the random variables

$$H_N(f) = \sum_{j=1}^{N} W_j u_j(f), \tag{2.5.1}$$

to an infinitely divisible law. This result is elementary since it only uses the fact that the random variables W_j's are independent and Poisson.

Lemma 2.5.1. *Under the assumption* (2.2.3), *i.e*

$$\sum_{j=1}^{N} jR_j^2 \in (0, \infty), \tag{2.5.2}$$

the distribution μ_N of $H_N - \mathbb{E}[H_N]$ converges weakly to the distribution $\mu_{f,\theta}$ defined by (2.2.4).

Proof. The Fourier transform of $H_N - \mathbb{E}[H_N]$ is easy to compute, indeed:

$$\begin{aligned} \log \widehat{\mu}_N(t) = \log \mathbb{E}\left[e^{it(H_N - \mathbb{E}[H_N])}\right] &= \log \prod_{j=1}^{N} \mathbb{E}\left[\exp\left(itu_j\left(W_j - \frac{\theta}{j}\right)\right)\right] \\ &= \sum_{j=1}^{N} \frac{\theta}{j}(e^{itu_j} - itu_j - 1). \end{aligned} \tag{2.5.3}$$

Obviously, for $|t| \leq T$,

$$\left|\frac{\theta}{j}\left(e^{itu_j} - itu_j - 1\right)\right| \leq \frac{\theta}{j}\frac{t^2 u_j^2}{2} \leq \theta \frac{T^2}{2}\frac{u_j^2}{j}. \tag{2.5.4}$$

By 2.1.9, $\log \widehat{\mu}_N(t)$ converges absolutely uniformly and its limit

$$\psi(t) = \sum_{j=1}^{\infty} \frac{\theta}{j}(e^{itu_j} - itu_j - 1) \tag{2.5.5}$$

is continuous. By Lévy's Theorem, $\exp(\psi(t))$ is the Fourier transform of the probability measure $\mu_{f,\theta}$ and μ_N converges in distribution to $\mu_{f,\theta}$ as N goes to infinity. $\square$

Obviously, $\mu_{f,\theta}$ is an infinitely divisible distribution and its Lévy-Khintchine representation is easy to write. We recall that an infinitely divisible distribution μ has Lévy-Khintchine representation (a, M, σ^2) if its Fourier transform is given by

$$\widehat{\mu}(t) = \exp\left(\int \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x) + iat - \frac{1}{2}\sigma^2 t^2\right), \tag{2.5.6}$$

where $a \in \mathbb{R}$, $\sigma > 0$ and M is an admissible Levy measure, i.e.

$$\int \frac{x^2}{1+x^2} dM(x) < \infty.$$

The distribution $\mu_{f,\theta}$ in Lemma 2.5.1 has therefore a Lévy-Khintchine representation $(a, \theta M, 0)$ with

$$a = \int \left(\frac{x}{1+x^2} - x\right) dM(x) = \sum_{j=1}^{\infty} \lambda_j \left(\frac{u_j}{1+u_j^2} - u_j\right) \tag{2.5.7}$$

and

$$M = \sum_{j=1}^{\infty} \frac{1}{j} \delta_{u_j}. \tag{2.5.8}$$

It is easy to see that the assumption 2.1.9 implies that $\int x^2 dM(x) < \infty$ so that M is admissible.

2.5.2 Proof of Theorem 2.2.3

We proceed now to the proof of Theorem 2.2.3, by using the Feller coupling bounds proved in Section 4.

We first prove the first and second statements of Theorem 2.2.3. Under the assumption that $\sum_{j=1}^{\infty} jR_J^2 < \infty$, we have seen that the sequence $|u_j|$ converges in $(C,1)$ sense to zero. Moreover, if $\theta < 1$, we assumed in Theorem 2.2.3 that the sequence $|u_j|$ converges in (C,θ) sense to zero. Thus, we know that the assumption of Theorem 2.4.12 is satisfied, and thus that,

$$\lim_{N\to\infty} \mathbb{E}|G_N - H_N| = 0. \tag{2.5.9}$$

Using now Lemma 2.5.1, we have proved that $G_N - \mathbb{E}(G_N)$ converges in distribution to $\mu_{f,\theta}$ defined by (2.2.4). But, by the basic identity (2.1.8), we know that $I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]$ has the same distribution as $G_N(f) - \mathbb{E}[G_N(f)]$. This proves the first two statements of Theorem 2.2.3.

The third statement is simple. Indeed, by (2.1.8) and by (2.5.9),

$$\mathbb{E}[I_{\sigma,N}(f)] = N\int_0^1 f(x)dx + \mathbb{E}(G_N) = N\int_0^1 f(x)dx + \mathbb{E}(H_N) + o(1). \tag{2.5.10}$$

In order to complete the proof, it suffices to mention that the expectation of H_N is easy to compute:

$$\mathbb{E}(H_N) = \theta \sum_{j=1}^{N} \frac{u_j}{j} = \theta \sum_{j=1}^{N} R_j. \tag{2.5.11}$$

This proves the third statement of Theorem 2.2.3.

The proof of the fourth statement follows a similar pattern. Again, by (2.1.8),

$$\mathrm{Var}[I_{\sigma,N}(f)] = \mathrm{Var}(G_N). \tag{2.5.12}$$

But if one also assumes, as in the fourth item of Theorem 2.2.3, that the sequence (u_j^2) converges in $(C, 1\wedge\theta)$ sense to zero, then by 2.4.13 we know that

$$\lim_{N\to\infty} \mathbb{E}((G_N - H_N)^2) = 0. \tag{2.5.13}$$

This, and 2.5.9, imply that

$$\mathrm{Var}(G_N) = \mathrm{Var}(H_N) + o(1) \tag{2.5.14}$$

In order to complete the proof, it suffices to compute the variance of H_N:

$$\mathrm{Var}(H_N) = \theta \sum_{j=1}^{N} \frac{u_j^2}{j} = \theta \sum_{j=1}^{N} jR_j^2 \tag{2.5.15}$$

This proves the fourth statement and completes the proof of of Theorem 2.2.3.

2.5.3 Proof of Theorem 2.1.1

We show here how Theorem 2.2.3 implies Theorem 2.1.1.
We will need the following simple facts.

Lemma 2.5.2. *(i) The assumption $\sum_{j=1}^{\infty} jR_j(f)^2 < \infty$ implies the $(C,1)$ convergence of the sequence $(|u_j(f)|)_{j\geq 1}$ to zero.*

(ii) If one assumes that $\sum_{j=1}^{\infty} jR_j(f)^2 < \infty$ and that the function f is of bounded variation, then the sequence $(|u_j(f)|)_{j\geq 1}$ converges in (C,θ) sense to zero, for any $\theta > 0$.

(iii) If one assumes that $\sum_{j=1}^{\infty} jR_j(f)^2 < \infty$ and that the function f is of bounded variation, then the sequence $(u_j(f)^2)_{j\geq 1}$ converges in (C,θ) sense to zero, for any $\theta > 0$.

Proof. The first item is well known (see statement (a), p. 79 of [85], Volume 1). It is a consequence of the simple application of the Cauchy-Schwarz inequality

$$|\frac{1}{N}\sum_{j=1}^{N} u_j| \leq \frac{1}{N}(\sum_{j=1}^{N} jR_j^2)^{\frac{1}{2}}(\sum_{j=1}^{N} j)^{\frac{1}{2}} \leq (\sum_{j=1}^{\infty} jR_j^2)^{\frac{1}{2}}. \tag{2.5.16}$$

So that

$$\limsup_{N\to\infty} |\frac{1}{N}\sum_{j=1}^{N} u_j| \leq (\sum_{j=1}^{\infty} jR_j^2)^{\frac{1}{2}}. \tag{2.5.17}$$

But the LHS of (2.5.17) does not depend on the initial k values of the sequence u_j. By setting these k values to zero, and by taking k large enough, we can then make the RHS as small as we want. This implies that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} u_j = 0. \tag{2.5.18}$$

This is the $(C,1)$ convergence to zero, claimed in item (i).
In order to prove the item(ii), we need the following observation.

Lemma 2.5.3. *If the function f is of bounded variation, then*

$$|R_j(f)| \leq \frac{TV(f)}{j}, \tag{2.5.19}$$

where $TV(f)$ denotes the total variation of f.

Proof. Since f is of bounded variation, it can be written as a difference of two non-decreasing functions

$$f = f^+ - f^-. \tag{2.5.20}$$

Using (4.1.9),

$$R_j(f^+) = \sum_{k=0}^{j-1} \int_{\frac{k}{j}}^{\frac{k+1}{j}} (f^+(\frac{k}{j}) - f^+(x))dx. \tag{2.5.21}$$

So that

$$|R_j(f^+)| \leq \frac{1}{j} \sum_{k=0}^{j-1} \left(f^+(\frac{k+1}{j}) - f^+(\frac{k}{j}) \right) \leq \frac{1}{j} TV(f^+). \tag{2.5.22}$$

Using the same argument for f^- gives the result of Lemma 2.5.3. □

So, this shows that the sequence $(u_j(f))_{j\geq 1}$ is bounded, when f is of bounded variation. Now, using the item (i) of this Lemma and item (iii) of Lemma 2.2.2, we see that the sequence $(|u_j(f)|)_{j\geq 1}$ and thus, $u_j(f)$ converges in (C, θ) sense to zero, for any value of $\theta > 0$.
The last item is trivial, since the sequence $u(f) = (u_j(f))_{j\geq 1}$ is bounded, say by the constant C. Indeed, then the Cesaro means of the sequence $u(f)^2 = (u_j(f)^2)_{j\geq 1}$ are bounded, for any $\theta > 0$ by

$$\sigma_N^\theta(u(f)^2) \leq C\sigma_N^\theta(|u(f)|). \tag{2.5.23}$$

This implies the (C, θ) convergence of the sequence $u(f)^2$. □

Thus, Lemma 2.5.2 shows that Theorem 2.2.3 implies Theorem 2.1.1. Indeed, the general assumptions needed in Theorem 2.2.3 about the Cesaro convergence of the sequences $u(f)$ and $u(f)^2$ are satisfied by Lemma 2.5.2.

2.6 Proofs of Theorem 2.2.4 and Theorem 2.1.4

2.6.1 A simple Gaussian convergence result for series of Poisson random variables

We give here a result of convergence in distribution for the random variables

$$H_N(f) = \sum_{j=1}^{N} W_j u_j(f) \tag{2.6.1}$$

to a Gaussian law, once centered and normalized. This result is again elementary since it only uses the fact that the random variables W_j's are independent and Poisson.
Here, we assume as in Theorem 2.2.4 that

$$\sum_{j=1}^{\infty} jR_j(f)^2 = \infty \tag{2.6.2}$$

and that

$$\max_{1\leq j\leq N} |u_j| = \max_{1\leq j\leq N} |jR_j| = o((\sum_{j=1}^{N} jR_j(f)^2)^{\frac{1}{2}}). \tag{2.6.3}$$

Let us denote by

$$\eta_N^2 = \mathrm{Var}(H_N) = \theta \sum_{j=1}^{N} jR_j(f)^2. \tag{2.6.4}$$

Lemma 2.6.1. *Under these assumptions, the distribution of*

$$\frac{H_N - \mathbb{E}[H_N]}{\eta_N}$$

converges weakly to $\mathcal{N}(0,1)$ *as* $N \to \infty$.

Proof. Write $\widetilde{H}_N$ for

$$\frac{H_N - \mathbb{E}[H_N]}{\eta_N} = \frac{H_N - \mathbb{E}[H_N]}{\sqrt{\mathrm{Var}[H_N]}} = \sum_{j=1}^{N} \frac{u_j(W_j - (\theta/j))}{\eta_N}, \tag{2.6.5}$$

then

$$\log \mathbb{E}\left[e^{it\widetilde{H}_N}\right] = \sum_{j=1}^{N} \frac{\theta}{j}(e^{itu_j/\eta_N} - itu_j/\eta_N - 1), \tag{2.6.6}$$

which gives the distribution of $\widetilde{H}_N$ in its Lévy-Khintchine representation $(a_N, \theta M_N, \sigma_N^2)$, with

$$a_N = \sum_{j=1}^{N} \left(\frac{\theta}{j(\eta_N^2 + u_j^2)} \left(\frac{-u_j^3}{\eta_N} \right) \right), \tag{2.6.7}$$

$$M_N = \sum_{j=1}^{N} \frac{1}{j} \delta_{u_j/\eta_N} \tag{2.6.8}$$

and $\sigma_N = 0$.
We continue this proof by applying the Lévy-Khintchine Convergence Theorem ([74], p. 62):
Consider a bounded continuous function f such that $f(x) = 0$ for $|x| < \delta$, then

$$\int f dM_N = \sum_{j=1}^{N} \frac{\theta}{j} f\left(\frac{u_j}{\eta_N}\right) \mathbb{1}_{\left|\frac{u_j}{\eta_N}\right| > \delta}. \tag{2.6.9}$$

Under the assumption (2.6.3), $\int f dM_N = 0$ for N large enough, so that

$$\lim_{N\to\infty} \int f dM_N = \int f dM = 0. \tag{2.6.10}$$

Again, using the assumption (2.6.3), we have that for any $\ell > 0$,

$$\int_{-\ell}^{\ell} x^2 dM_N + \sigma_N^2 = \sum_{j=1}^{N} \frac{\theta}{j} \frac{u_j^2}{\eta_N^2} \mathbb{1}_{|u_j|<\ell\eta_N}. \tag{2.6.11}$$

So for N large enough, $\int_{-\ell}^{\ell} x^2 dM_N = 1$ and so

$$\lim_{N\to\infty} \int_{-\ell}^{\ell} x^2 dM_N = 1. \tag{2.6.12}$$

Moreover, for every N define

$$\epsilon_N = \frac{\max_{1\le j\le N} |u_j|}{\eta_N}, \tag{2.6.13}$$

then we can bound a_N above by

$$|a_N| \le \sum_{j=1}^{N} \left(\frac{\theta/j}{\eta_N^2 + u_j^2} (\epsilon_N u_j^2) \right) \le \frac{\epsilon_N}{\eta_N^2} \sum_{j=1}^{N} \frac{\theta}{j} u_j^2 = \epsilon_N. \tag{2.6.14}$$

By the assumption (2.6.3), we see that

$$\lim_{N\to\infty} a_N = 0. \tag{2.6.15}$$

By (2.6.10), (2.6.12), (2.6.15) and using Theorem 3.21, p. 62 in [74], we see that $\widetilde{H}_N$ converges in distribution to the infinitely divisible distribution with Lévy-Khintchine representation $(a, M, \sigma^2) = (0, 0, 1)$, i.e. to the standard normal Gaussian $\mathcal{N}(0,1)$.

□

2.6.2 Proof of Theorem 2.2.4

We proceed now to the proof of Theorem 2.2.4, by using the Feller coupling bounds proved in Section 4. Again, we assume here, as in Theorem 2.2.4, that

$$\sum_{j=1}^{\infty} jR_j(f)^2 = \infty \tag{2.6.16}$$

and that

$$\max_{1\le j\le N} |jR_j| = o((\sum_{j=1}^{N} jR_j(f)^2)^{\frac{1}{2}}) \tag{2.6.17}$$

which can be rewritten

$$\max_{1\le j\le N} |u_j| = o(\eta_N) \tag{2.6.18}$$

By Lemma 2.4.2, we know that

$$\mathbb{E}(|G_N - H_N|) \leq C(\theta) \max_{1\leq j\leq N} |u_j| = o(\eta_N) \tag{2.6.19}$$

Again, denote by $\widetilde{H}_N := \frac{H_N - \mathbb{E}[H_N]}{\eta_N}$ and $\widetilde{G}_N := \frac{G_N - \mathbb{E}[G_N]}{\eta_N}$. Then obviously,

$$\mathbb{E}\left[|\widetilde{G}_N - \widetilde{H}_N)|\right] = o(1), \tag{2.6.20}$$

which, together with the convergence result (Lemma 2.6.1) for H_N proves that $\widetilde{G}_N$ converges in distribution to a standard Gaussian law N(0,1).

Moreover, this also proves that

$$\mathbb{E}[G_N] = \mathbb{E}[H_N] + o(\eta_N) = \theta \sum_{j=1}^{N} R_j + o(\eta_N). \tag{2.6.21}$$

And, by Lemma 2.4.8, we also know that

$$\mathbb{E}((G_N - H_N)^2) \leq C(\theta) \max_{1\leq j\leq N} u_j^2 = o(\eta_N^2). \tag{2.6.22}$$

Then obviously,

$$\mathbb{E}\left[(\widetilde{G}_N - \widetilde{H}_N)^2\right] \leq 2\left(\mathbb{E}\left[(G_N - H_N)^2\right] + \mathbb{E}\left[G_N - H_N\right]^2\right) = o(\eta_N^2). \tag{2.6.23}$$

So that

$$|\sqrt{\mathrm{Var}(G_N)} - \sqrt{\mathrm{Var}(H_N)}| = o(\eta_N) \tag{2.6.24}$$

and thus,

$$\mathrm{Var}(G_N) \sim \mathrm{Var}(H_N) = \eta_N^2. \tag{2.6.25}$$

From these three results, we get that

$$\frac{G_N - \mathbb{E}[G_N]}{\sqrt{\mathrm{Var}(G_N)}}$$

converges in distribution to a standard Gaussian $\mathcal{N}(0,1)$. But $I_{\sigma,N}(f) - \mathbb{E}[I_{\sigma,N}(f)]$ has the same distribution as $G_N - \mathbb{E}[G_N]$ and thus,

$$\frac{I_{\sigma,N} - \mathbb{E}[I_{\sigma,N}]}{\sqrt{\mathrm{Var}(I_{\sigma,N})}}$$

converges also in distribution to a $\mathcal{N}(0,1)$ distribution. We thus have proved the first statement of Theorem 2.2.4. Moreover, we have that

$$\mathbb{E}[I_N] = \int_0^1 f(x)dx + \mathbb{E}[G_N] = \int_0^1 f(x)dx + \theta \sum_{j=1}^{N} R_j + o(\eta_N), \tag{2.6.26}$$

which is the second statement of Theorem 2.2.4. Finally,

$$\mathrm{Var}(I_{\sigma,N}) = \mathrm{Var}(G_N) \sim \mathrm{Var}(H_N) = \eta_N^2, \tag{2.6.27}$$

which is the third statement. We have completed the proof of Theorem 2.2.4.

2.6.3 Proof of Theorem 2.1.4

We prove here how Theorem 2.2.4 implies Theorem 2.1.4. In Theorem 2.1.4 we assumed that f is of bounded variation, which implies, as we have seen, that $u_j(f) = O(1)$, and thus, that

$$\max_{1\le j\le N} |u_j| = o(\eta_N) \tag{2.6.28}$$

since the sequence η_N is assumed to diverge. This proves that the hypothesis of Theorem 2.1.4 are satisfied under those of Theorem 2.2.4. Thus we get that the conclusions of Theorem 2.2.4 are valid. They are almost exactly the same as the conclusions of Theorem 2.1.4. The only thing left to prove is the item (ii). But using Lemma 2.4.2, 2.5.11, and the fact that $u_j(f) = jR_j(f) = O(1)$, we have that

$$\mathbb{E}\left[I_{\sigma,N}(f)\right] = N\int_0^1 f(x)dx + \sum_{j=1}^N R_j(f) + O(1). \tag{2.6.29}$$

The bound

$$\sum_{j=1}^N R_j(f) = O(\log N) \tag{2.6.30}$$

is trivial since again $R_j = O(\frac{1}{j})$. With this we have derived Theorem 2.1.4 from Theorem 2.2.4.

2.7 The expectation and the variance

For the sake of completeness, we give here the explicit expressions for the expectation and the variance of $I_{\sigma,N}$, when σ is chosen from $\mathcal{S}_N$ by the Ewens distribution with parameter θ. The basic computations for the expectation and the variance of the cycle counts can be simply derived by the following formula established by Watterson [75] (see Arratia, Barbour and Tavaré [3], (4.7), p. 68):
For every $b \ge 1$, $(r_1, \dots, r_b) \ge 0$,

$$\mathbb{E}\left[\prod_{j=1}^b \alpha_j^{[r_j]}\right] = \mathbb{1}_{m\le N}\binom{N-m+\gamma}{N-m}\binom{N+\gamma}{N}^{-1}\prod_{j=1}^b\left(\frac{\theta}{j}\right)^{r_j}, \tag{2.7.1}$$

where $m = \sum_{j=1}^b jr_j$, $x^{[r]} = x(x-1)\dots(x-r+1)$ and $\gamma = \theta - 1$.
Thus, the mean and the variance of $I_{\sigma,N}$ can be easily computed.

Lemma 2.7.1.

$$\begin{aligned}\mathbb{E}[I_{\sigma,N}(f)] &= N\int_0^1 f(x)dx + \sum_{j=1}^N \mathbb{E}[\alpha_j(\sigma)]u_j(f)\\ &= N\int_0^1 f(x)dx + \theta\sum_{j=1}^N \Psi_N(j)R_j(f)\end{aligned} \tag{2.7.2}$$

Proof. From the general formula (2.7.1), we easily see that for any $j \geq 1$ and any $\theta > 0$,

$$\mathbb{E}_\theta[\alpha_j] = \frac{\theta}{j} \Psi_N(j) \mathbb{1}_{j \leq N}. \tag{2.7.3}$$

Thus, (2.7.2) follows immediately. □

The variance of $I_{\sigma,N}$ is given by the following lemma:

Lemma 2.7.2.

$$\begin{aligned}
\operatorname{Var}[I_{\sigma,N}(f)] &= \operatorname{Var}\left(\sum_{j=1}^{N} \alpha_j u_j(f)\right) \\
&= \theta \sum_{j=1} j R_j^2 \Psi_N(j) + \theta^2 \sum_{j,j' \leq N} R_j R_{j'} \left(\Psi_N(j+j') \mathbb{1}_{j+j' \leq N} - \Psi_N(j)\Psi_N(j')\right) \\
&= \eta_N^2 + \theta \sum_{j=1} j R_j^2 (\Psi_N(j) - 1) \\
&\quad + \theta^2 \sum_{j,j' \leq N} R_j R_{j'} \left(\Psi_N(j+j') \mathbb{1}_{j+j' \leq N} - \Psi_N(j)\Psi_N(j')\right)
\end{aligned} \tag{2.7.4}$$

Proof. Again, from the general formula (2.7.1), we easily see that for any $j \geq 1$ and any $\theta > 0$,

$$\mathbb{E}_\theta[\alpha_j \alpha_{j'}] = \frac{\theta^2}{jj'} \Psi_N(j+j') \mathbb{1}_{j+j' \leq N} \tag{2.7.5}$$

and

$$\mathbb{E}_\theta[\alpha_j^2] = \frac{\theta^2}{j^2} \Psi_N(2j) \mathbb{1}_{j \leq N/2} + \frac{\theta}{j} \Psi_N(j) \mathbb{1}_{j \leq N}. \tag{2.7.6}$$

The variance of α_j is therefore given by

$$\operatorname{Var}_\theta[\alpha_j] = \frac{\theta}{j} \Psi_N(j) \mathbb{1}_{j \leq N} + \frac{\theta^2}{j^2} \Psi_N(2j) \mathbb{1}_{j \leq N/2} - \frac{\theta^2}{j^2} \Psi_N(j)^2 \mathbb{1}_{j \leq N} \tag{2.7.7}$$

and the covariance by

$$\operatorname{Cov}_\theta[\alpha_j, \alpha_{j'}] = \frac{\theta^2}{jj'} \Psi_N(j+j') \mathbb{1}_{j+j' \leq N} - \frac{\theta^2}{jj'} \Psi_N(j) \Psi_N(j') \mathbb{1}_{j \leq N} \mathbb{1}_{j' \leq N}, \tag{2.7.8}$$

for $j \neq j'$. Then, the variance of $I_{\sigma,N}$ given in (2.7.4) follows immediately. □

Remark 2.7.2.1. *The case where $\theta = 1$ is particularly simple. Indeed then*

$$\mathbb{E}[I_{\sigma,N}(f)] = N \int_0^1 f(x)dx + \theta \sum_{j=1}^{N} R_j(f)$$

and

$$\operatorname{Var}[I_{\sigma,N}(f)] = \eta_N^2.$$

Thus the asymptotic formulae we give in this work are then exact. For general values of $\theta > 0$, it is possible to derive these asymptotic expressions directly from the explicit formulae given in this Section, without using the bounds on the Feller coupling, but this is not a trivial matter, in particular when $\theta < 1$.

Chapter 3

Fluctuations of the characteristic polynomial

We consider the logarithm of the characteristic polynomial of random permutation matrices, evaluated on a finite set of different points. The permutations are chosen with respect to the Ewens distribution on the symmetric group. We show that the behavior at different points is independent in the limit and are asymptotically normal. Our methods enable us to study more general matrices, closely related to permutation matrices, and multiplicative class functions.

3.1 Introduction

The characteristic polynomial of a random matrix is a well studied object in Random Matrix Theory (RMT) (see for example [13], [17], [44], [40], [49], [42], [82], [83]). By a central limit theorem result of Keating and Snaith for $n \times n$ CUE matrices [49], the imaginary and the real part of the logarithm of the characteristic polynomial converge jointly in law to independent standard normal distributed random variables, after normalizing by $\sqrt{(1/2)\log n}$. Hughes, Keating and O'Connell refined this result in [44]: evaluating the logarithm of the characteristic polynomial, normalized by $\sqrt{(1/2)\log n}$, for a discrete set of points on the unit circle, this leads to a collection of i.i.d. standard (complex) normal random variables.

In [40], Hambly, Keevash, O'Connell and Stark give a Gaussian limit for the logarithm of the characteristic polynomial of random permutation matrices under uniform measure on the symmetric group. This result has been extended by Zeindler in [83] to the Ewens distribution on the symmetric group and to the logarithm of multiplicative class functions, introduced in [22].

In this paper, we will generalize the results in [40] and [83] in two ways. First, we follow the spirit of [44] by considering the behavior of the logarithm of the characteristic polynomial of a random permutation matrix at different points $x_1, \dots, x_d$. Second, we state CLT's for the logarithm of characteristic polynomials for matrix groups related to permutation matrices, such as some Weyl groups [22, section 7] and of the wreath product $\mathbb{T} \wr S_n$ [78] .
In particular, we consider $n \times n$-matrices $M = (M_{ij})_{1 \le i,j \le n}$ of the following form: For a permutation $\sigma \in S_n$ and a complex valued random variable z,

$$M_{ij}(\sigma, z) := z_i \delta_{i,\sigma(j)}, \tag{3.1.1}$$

where z_i is a family of i.i.d. random variables s.t. $z_i \stackrel{d}{=} z$, z_i independent of σ. Here, σ is chosen with respect to the Ewens distribution, i.e.

$$\mathbb{P}_\theta\left[\sigma\right] := \frac{\theta^{l_\sigma}}{\theta(\theta+1)\dots(\theta+n-1)}, \tag{3.1.2}$$

for fixed parameter $\theta > 0$ and l_σ being the total number of cycles of σ. The *Ewens measure* or *Ewens distribution* is a well-known measure on the the symmetric group S_n, appearing for example in population genetics [36]. It can be viewed as a generalization of the uniform distribution (i.e $\mathbb{P}\left[A\right] = \frac{|A|}{n!}$) and has an additional weight depending on the total number of cycles. The case $\theta = 1$ corresponds to the uniform measure. Matrices $M_{\sigma,z}$ of the form (3.1.1) can be viewed as generalized permutation matrices $M_\sigma = M_{\sigma,1}$, where the 1-entries are replaced by i.i.d. random variables. Also, it is easy to see that elements of the wreath product $\mathbb{T} \wr S_n$ for $z \in \mathbb{T}^n$ (see [78] and [22, section 4.2]) or elements of some Weyl groups (treated in [22, section 7]) are of the form (3.1.1). In this paper, we will not give any more details about wreath products and Weyl groups, since we do not use group structures.

We define the function $Z_{n,z}(x)$ by

$$Z_{n,z}(x) := \det(I - x^{-1}M_{\sigma,z}), \quad x \in \mathbb{C}. \tag{3.1.3}$$

Then, the characteristic polynomial of $M_{\sigma,z}$ has the same zeros as $Z_{n,z}(x)$. We will study the characteristic polynomial by identifying it with $Z_{n,z}(x)$, following the convention of [22], [82] or [83].
By using that the random variables z_i, $1 \leq i \leq n$ are i.i.d., a simple computation shows the following equality in law (see [22], Lemma 4.2):

$$Z_{n,z}(x) \stackrel{d}{=} \prod_{m=1}^{n} \prod_{k=1}^{C_m} (1 - x^{-m} T_{m,k}), \tag{3.1.4}$$

where C_m denotes the number of cycles of length m in σ and $(T_{m,k})_{1 \leq m,k \leq \infty}$ is a family of independent random variables, independent of $\sigma \in S_n$, such that

$$T_{m,k} \stackrel{d}{=} \prod_{j=1}^{m} z_j. \tag{3.1.5}$$

Note that the characteristic polynomial $Z_{n,z}(x)$ of $M_{\sigma,z}$ depends strongly on the random variables C_m $(1 \leq m \leq n)$. The distribution of $(C_1, C_2, \cdots, C_n)$ with respect to the Ewens distribution with parameter θ was first derived by Ewens (1972), [36]. It can be computed, using the inclusion-exclusion formula, [3, chapter 4, (4.7)].

We are interested in the asymptotic behavior of the logarithm of (3.1.3) and therefore, we will study the characteristic polynomial of $M_{\sigma,z}$ in terms of (3.1.4), by choosing the branch of logarithm in a suitable way. In view of (3.1.4), it is natural to choose it as follows:

Definition 3.1.1. *Let $x = e^{2\pi i\varphi} \in \mathbb{T}$ be a fixed number and z a $\mathbb{T}$–valued random variable. Furthermore, let $(z_{m,k})_{m,k=1}^{\infty}$ and $(T_{m,k})_{m,k=1}^{\infty}$ be two sequences of independent random variables, independent of $\sigma \in S_n$ with*

$$z_{m,k} \stackrel{d}{=} z \quad \text{and} \quad T_{m,k} \stackrel{d}{=} \prod_{j=1}^{m} z_{j,k}. \tag{3.1.6}$$

We then set

$$\log\bigl(Z_{n,z}(x)\bigr) := \sum_{m=1}^{n} \sum_{k=1}^{C_m} \log(1 - x^{-m} T_{m,k}), \tag{3.1.7}$$

where we use for $\log(.)$ *the principal branch of logarithm. We will deal with negative values as follows:* $\log(-y) = \log y + i\pi$, $y \in \mathbb{R}_+$. *Note, that it is not necessary to specify the logarithm at* 0, *since we will deal only with cases where this occurs with probability* 0.

In this paper, we show that under various conditions, $\log Z_{n,z}(x)$ converges to a complex standard Gaussian distributed random variable after normalization and the behavior at different points is independent in the limit. Moreover, the normalization by $\sqrt{(\pi^2/12)\theta \log n}$ is independent of the random variable z. This covers the result in [40] for $\theta = 1$ and z being deterministic equal to 1. We state this more precisely:

Proposition 3.1.1. *Let S_n be endowed with the Ewens distribution with parameter θ, z a $\mathbb{T}$-valued random variable and $x \in \mathbb{T}$ be not a root of unity, i.e. $x^m \neq 1$ for all $m \in \mathbb{Z}$.*
Suppose that z is uniformly distributed. Then, as $n \to \infty$,

$$\frac{\mathrm{Re}\,\bigl(\log\bigl(Z_{n,z}(x)\bigr)\bigr)}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \xrightarrow{d} N_R \quad \text{and} \tag{3.1.8}$$

$$\frac{\mathrm{Im}\,\bigl(\log\bigl(Z_{n,z}(x)\bigr)\bigr)}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \xrightarrow{d} N_I, \tag{3.1.9}$$

with $N_R, N_I \sim \mathcal{N}(0,1)$.

In Proposition 3.1.1 $\mathrm{Re}\,\bigl(\log\bigl(Z_{n,z}(x)\bigr)\bigr)$ and $\mathrm{Im}\,\bigl(\log\bigl(Z_{n,z}(x)\bigr)\bigr)$ are converging to normal random variables without centering. This is due to that the expectation is $o(\sqrt{\log n})$. This will become more clear in the proof (see Section 3.4.1).
Furthermore, we state a CLT for $\log Z_{n,z}(x)$, evaluated on a finite set of different points $\{x_1, \ldots, x_d\}$.

Proposition 3.1.2. *Let S_n be endowed with the Ewens distribution with parameter θ, $\overline{z} = (z_1, \ldots, z_d)$ be a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i\varphi_1}, \ldots, x_d = e^{2\pi i\varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \ldots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*

Suppose that $z_1, \dots, z_d$ *are uniformly distributed and independent. Then we have, as* $n \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \begin{pmatrix} \log(Z_{n,z_1}(x_1)) \\ \vdots \\ \log(Z_{n,z_d}(x_d)) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\mathrm{Re}(N_1), \dots, \mathrm{Re}(N_d), \mathrm{Im}(N_1), \dots, \mathrm{Im}(N_d)$ *independent standard normal distributed random variables.*

Note that $z_1, \dots, z_d$ are not equal to the family $(z_i)_{1 \le i \le n}$ of i.i.d. random variables in (3.1.1). In fact, we deal here with d different families of i.i.d. random variables, where the distributions are given by $z_1, \dots, z_d$. We will treat this more carefully in Section 3.4.2.

Proposition 3.1.2 shows that the characteristic polynomial of the random matrices $M_{\sigma,z}$ follows the tradition of matrices in the CUE, if evaluated at different points, due to the result by [44]. Moreover, the proof of Proposition 3.1.2 can also be used for regular random permutation matrices, i.e. $M_{\sigma,1}$, but requires further assumptions on the points $x_1, \dots, x_d$. We state this more precisely:

Proposition 3.1.3. *Let* S_n *be endowed with the Ewens distribution with parameter* θ *and* $x_1 = e^{2\pi i \varphi_1}, \dots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ *be pairwise of finite type (see Definition 3.2.18).*
Then, as $n \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \begin{pmatrix} \log(Z_{n,1}(x_1)) \\ \vdots \\ \log(Z_{n,1}(x_d)) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\mathrm{Re}(N_1), \dots, \mathrm{Re}(N_d), \mathrm{Im}(N_1), \dots, \mathrm{Im}(N_d)$ *independent standard normal distributed random variables.*

In fact, our methods allow us to prove much more. First, we are able to relax the conditions in the Propositions 3.1.1, 3.1.2 and 3.1.3 above. Also, these results on $\log Z_{n,z}(x)$ follow as corollaries of much more general statements (see Section 3.4). Indeed, the methods allow us to prove CLT's for multiplicative class functions. Multiplicative class functions have been studied by Dehaye and Dehaye-Zeindler, [22], [83].

Following [22], we present here two different types of multiplicative class functions.

Definition 3.1.2. *Let* z *be a complex valued random variable and* $f : \mathbb{C} \to \mathbb{C}$ *be given. Write for* $\sigma \in S_n$

$$W^1(f)(x) = W_z^{1,n}(f)(x)(\sigma) := \prod_{m=1}^{n} \prod_{k=1}^{C_m} f(z_m x^m), \tag{3.1.10}$$

where $z_m \stackrel{d}{=} z$*,* z_m *i.i.d. and independent of* σ*. This defines the* first multiplicative class function associated to f.

The second multiplicative class function is directly motivated by the expression (3.1.4) and is a slightly modified form of (3.1.10).

Definition 3.1.3. *Let* z *be a complex valued random variable and* $f : \mathbb{C} \to \mathbb{C}$ *be given. Write for* $\sigma \in S_n$

$$W^2(f)(x) = W_z^{2,n}(f)(x)(\sigma) := \prod_{m=1}^{n} \prod_{k=1}^{C_m} f\left(x^m T_{m,k}\right), \tag{3.1.11}$$

where $T_{m,k}$ *is a family of independent random variables,* $T_{m,k} \stackrel{d}{=} \prod_{j=1}^{m} z_j$ *and* $z_j \stackrel{d}{=} z$*, for any* $1 \leq j \leq n$*. This defines the* second multiplicative class function associated to f.

It is obvious from (3.1.4) and (3.1.11) that $Z_{n,z}(x)$ is the special case $f(x) = 1 - x^{-1}$ of $W^2(f)(x)$. This explains, why results on the second multiplicative class function cover in general results on $\log Z_{n,z}(x)$.
We postpone the statements of the more general theorems on multiplicative class functions to Section 3.4.

For the proofs we will make use of similar tools as in [40] and [83]. These tools include the Feller Coupling, uniformly distributed sequences and Diophantine approximations.
The structure of this paper is as follows: In Section 3.2, we will give some background of the Feller Coupling. Moreover, we recall some basic facts on uniformly distributed sequences and Diophantine approximations. In Section 3.3, we state some auxiliary CLT's on the symmetric group, which we will use in Section 3.4 to prove our main results for the characteristic polynomials and more generally, for multiplicative class functions.

3.2 Preliminaries

3.2.1 The Feller coupling

The reason why we expand the characteristic polynomial of $M_{\sigma,z}$ in terms of the cycle counts of σ as given in (3.1.4) is the fact that the asymptotic behavior of the numbers of cycles with length m in σ, denoted by $(C_m)_{1 \leq m \leq n}$, has been well-studied, for example by [3] or [36]. In particular, they are in the limit independent Poisson random variables with mean θ/m, $m \geq 1$. To state this more precisely, we make use of the Feller coupling (see for instance [3], [36], [75]), which links the family of cycle lengths with the family of Poisson random variables.
Let D_i, for $i \geq 1$, be independent random variables s.t. for $\theta > 1$

$$\mathbb{P}[D_i = 1] = \frac{\theta}{\theta + i - 1}, \qquad \mathbb{P}[D_i = j] = \frac{1}{\theta + i - 1}, \quad 2 \leq j \leq i.$$

$D_i = 1$ corresponds to starting a new cycle $(i \ldots$, whereas we put i after $j-1$ in the given cycle for $D_i = j$ $(2 \leq j \leq N)$, proceeding in order $i = 1$, $i = 2$ until $i = n$. Then the sequence

$$D_1 D_2 \ldots D_n$$

produces a permutation in $\mathcal{S}_n$ under the Ewens distribution P_θ, defined in (3.1.2). There is even a one-to-one correspondence between S_n and sequences $(D_i)_{1 \leq i \leq n}$. For given permutation σ in ordered cycle type, we set $D_i = 1$ if i is the first number in a cycle and $D_i = j$ if $j-1$ is the next number in front of i in the increasing subsequence of the cycle.

Definition 3.2.1. *Let $\xi_i = 1_{\{D_i=1\}}$ be independent Bernoulli random variables for $i \geq 1$ with*

$$\mathbb{P}[\xi_i = 1] = \frac{\theta}{\theta + i - 1} \quad \text{and} \quad \mathbb{P}[\xi_i = 0] = \frac{i-1}{\theta + i - 1}.$$

Define $C_m^{(n)}(\xi)$ to be the number of m-spacings in $1\xi_2 \cdots \xi_n 1$ and $Y_m(\xi)$ to be the number of m-spacings in the limit sequence, i.e.

$$C_m^{(n)}(\xi) = \sum_{i=1}^{n-m} \xi_i (1 - \xi_{i+1}) \ldots (1 - \xi_{i+m-1}) \xi_{i+m} + \xi_{n-m+1} (1 - \xi_{n-m+2}) \ldots (1 - \xi_n) \tag{3.2.1}$$

and

$$Y_m(\xi) = \sum_{i=1}^{\infty} \xi_i (1 - \xi_{i+1}) \ldots (1 - \xi_{i+m-1}) \xi_{i+m}. \tag{3.2.2}$$

Then the following theorem holds (see [3, Chapter 4, p. 87] and [2, Theorem 2]).

Theorem 3.2.2. *Under the Ewens distribution, we have that*

- *The above-constructed $C_m^{(n)}(\xi)$ has the same distribution as the variable $C_m^{(n)} = C_m$, the number of cycles of length m in σ.*
- *$Y_m(\xi)$ is a.s. finite and Poisson distributed with $\mathbb{E}[Y_m(\xi)] = \frac{\theta}{m}$.*
- *All $Y_m(\xi)$ are independent.*
- *For any fixed $b \in \mathbb{N}$,*

$$\mathbb{P}\left[(C_1^{(n)}(\xi), \cdots, C_b^{(n)}(\xi)) \neq (Y_1(\xi), \cdots, Y_b(\xi))\right] \to 0 \ (n \to \infty).$$

Furthermore, the distance between $C_m^{(n)}(\xi)$ and $Y_m(\xi)$ can be bounded from above (see for example [2], p. 525). We will give here the following bound (see [10], p. 15):

Lemma 3.2.3. *For any $\theta > 0$ there exists a constant $K(\theta)$ depending on θ, such that for every $1 \leq m \leq n$,*

$$\mathbb{E}_\theta\left[\left|C_m^{(n)}(\xi) - Y_m(\xi)\right|\right] \leq \frac{K(\theta)}{n} + \frac{\theta}{n} \Psi_n(m), \tag{3.2.3}$$

where

$$\Psi_n := \binom{n-m+\theta-1}{n-m}\binom{n+\theta-1}{n}^{-1}. \tag{3.2.4}$$

Note that Ψ_n satisfies the following equality:

Lemma 3.2.4. *For each $\theta > 0$, there exist some constants K_1 and K_2 such that*

$$\Psi_n(m) \leq \begin{cases} K_1(1-\frac{m}{n})^{\theta-1} & \text{for } m < n, \\ K_2 n^{1-\theta} & m = n. \end{cases} \tag{3.2.5}$$

Proof. Let $\gamma = \theta - 1$. It is well known (see [85], p.77) that

$$\lim_{k\to\infty} \frac{1}{k^\gamma}\binom{\gamma+k}{k} = \frac{1}{\Gamma(\theta)}. \tag{3.2.6}$$

Moreover,

$$\binom{\gamma+k}{k} = \frac{k^\gamma}{\Gamma(\theta)}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{3.2.7}$$

Then, the case where $m = n$ is clear. Consider the case $m < n$. By (3.2.6), there exist numbers $0 < a < A$ for an integer K_0 depending on θ such that, for $k \geq K_0$,

$$\frac{ak^\gamma}{\Gamma(\theta)} \leq \binom{\gamma+k}{k} \leq \frac{Ak^\gamma}{\Gamma(\theta)}. \tag{3.2.8}$$

Let $n \geq K_0$ and $\delta_n n \geq K_0$ for a sequence δ_n with $\delta_n n \to \infty$ as $n \to \infty$. Let $m \leq (1-\delta_n)n$, then by (3.2.8) we have

$$\binom{\gamma+n-m}{n-m}\binom{\gamma+n}{n}^{-1} \leq \frac{A}{a}\left(\frac{n-m}{n}\right)^\gamma. \tag{3.2.9}$$

This proves the claim. □

3.2.2 Uniformly distributed sequences

We introduce in this section uniformly distributed sequences and some of their properties. Most of this section is well-known. The only new result is Theorem 3.2.13, which is an extension of the Koksma-Hlawka inequality. For the other proofs (and statements), see the books by Drmota and Tichy [26] and by Kuipers and Niederreiter [52].

We begin by giving the definition of uniformly distributed sequences.

Definition 3.2.5. *Let $\boldsymbol{\varphi} = \left(\varphi^{(m)}\right)_{m=1}^\infty$ be a sequence in $[0,1]^d$. For $\overline{\alpha} = (\alpha_1, \dots, \alpha_d) \in [0,1]^d$, we set*

$$A_n(\overline{\alpha}) = A_n(\overline{\alpha}, \boldsymbol{\varphi}) := \#\left\{1 \leq m \leq n; \varphi_m \in [0,\alpha_1] \times \cdots \times [0,\alpha_d]\right\}. \tag{3.2.10}$$

The sequence $\boldsymbol{\varphi}$ is called uniformly distributed in $[0,1]^d$ if we have

$$\lim_{n\to\infty}\left|\frac{A_n(\overline{\alpha})}{n} - \prod_{j=1}^d \alpha_j\right| = 0 \text{ for any } \overline{\alpha} \in [0,1]^d. \tag{3.2.11}$$

The following theorem shows that the name uniformly distributed is well chosen.

Theorem 3.2.6. *Let $h : [0,1]^d \to \mathbb{C}$ be a proper Riemann integrable function and $\boldsymbol{\varphi} = \left(\varphi^{(m)}\right)_{m\in\mathbb{N}}$ be a uniformly distributed sequence in $[0,1]^d$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} h(\varphi^{(m)}) = \int_{[0,1]^d} h(\overline{\phi})\, d\overline{\phi}, \tag{3.2.12}$$

where $d\overline{\phi}$ is the d-dimensional Lebesgue measure.

Proof. See [52, Theorem 6.1] □

Next, we introduce the discrepancy of a sequence $\boldsymbol{\varphi}$.

Definition 3.2.7. *Let $\boldsymbol{\varphi} = \left(\varphi^{(m)}\right)_{m=1}^{\infty}$ be a sequence in $[0,1]^d$. The $*-$discrepancy is defined as*

$$D_n^* = D_n^*(\boldsymbol{\varphi}) := \sup_{\overline{\alpha}\in[0,1]^d} \left| \frac{A_n(\overline{\alpha})}{n} - \prod_{j=1}^{d} \alpha_j \right|. \tag{3.2.13}$$

Remark There exists also a discrepancy D_n (without the $*$), which is equivalent to D_n^*, i.e. $D_n^* \leq D_n \leq 2^d D_n^*$. We need here only D_n^* and thus omit the definition of D_n.
By the following lemma, Theorem 3.2.6, the discrepancy and uniformly distributed sequences are closely related.

Lemma 3.2.8. *Let $\boldsymbol{\varphi} = \left(\varphi^{(m)}\right)_{m=1}^{\infty}$ be a sequence in $[0,1]^d$. The following statements are equivalent:*

1. *$\boldsymbol{\varphi}$ is uniformly distributed in $[0,1]^d$.*
2. $\lim_{n\to\infty} D_n^*(\boldsymbol{\varphi}) = 0$.
3. *Let $h : [0,1]^d \to \mathbb{C}$ be a proper Riemann integrable function. Then*

$$\frac{1}{n} \sum_{m=1}^{n} h(\varphi^{(m)}) \to \int_{[0,1]^d} h(\overline{\phi})\, d\overline{\phi} \quad \textit{for } n \to \infty.$$

The discrepancy allows us to estimate the rate of convergence in Theorem 3.2.6. To state this more precise, we introduce some notation.

Definition 3.2.9. *Let $h : [0,1]^d \to \mathbb{C}$ be a function. We call h of bounded variation in the sense of Hardy and Krause, if h is of bounded variation in the sense of Vitali and h restricted to each face F of dimension $1,\dots,d-1$ of $[0,1]^d$ is also of bounded variation in the sense of Vitali. We write $V(h|F)$ for the variation of h restricted to face F.*

Definition 3.2.10. *Let F be a face of $[0,1]^d$. We call a face F positive if there exists a sequence $j_1, \cdots, j_k$ in $\{1,\ldots,d\}$ s.t. $F = \bigcap_{m=1}^{k} \{s_{j_m} = 1\}$, with s_j, $1 \le j \le d$, being the canonical coordinates in $[0,1]^d$.*

Definition 3.2.11. *Let F be a face of $[0,1]^d$ and $\boldsymbol{\varphi}$ be sequence in $[0,1]^d$. Let $\pi_F(\boldsymbol{\varphi})$ be the projection of the sequence $\boldsymbol{\varphi}$ to the face F. We then write $D_n^*(F,\boldsymbol{\varphi})$ for the discrepancy of the projected sequence computed in the face F.*

We are now ready to state the following theorem:

Theorem 3.2.12 (Koksma-Hlawka inequality)**.** *Let $h : [0,1]^d \to \mathbb{C}$ be a function of bounded variation in the sense of Hardy and Krause. Let $\boldsymbol{\varphi} = \left(\varphi^{(m)}\right)_{m\in\mathbb{N}}$ be an arbitrary sequence in $[0,1]^d$. Then*

$$\left| \frac{1}{n} \sum_{m=1}^{n} h(\varphi^{(m)}) - \int_{[0,1]^d} h(\overline{\phi}) \, d\overline{\phi} \right| \le \sum_{k=1}^{d} \sum_{\substack{F \text{ positive} \\ dim(F)=k}} D_n^*(F,\boldsymbol{\varphi}) V(h|F) \tag{3.2.14}$$

Proof. See [52, Theorem 5.5]. $\square$

We will consider in this paper only functions of the form

$$h(\overline{\phi}) = h(\phi_1,\ldots,\phi_d) = \prod_{j=1}^{d} \log\bigl(f_j(e^{2\pi i \phi_j})\bigr), \tag{3.2.15}$$

with f_j being (piecewise) real analytic. In the context of the characteristic polynomial, we will choose $f_j(\phi_j) = |1 - e^{2\pi i \phi_j}|$. Unfortunately, we cannot apply Theorem 3.2.12 in this case, since $\log\bigl|1 - e^{2\pi i \phi_j}\bigr|$ is not of bounded variation. We thus reformulate Theorem 3.2.12. In order to do this, we follow the idea in [40] and [82] and replace $[0,1]^d$ by a slightly smaller set Q such that $\boldsymbol{\varphi} \subset Q$ and $h|Q$ is of bounded variation in the sense of Hardy and Krause.
We begin with the choice of Q. Considering (3.2.15), it is clear that the zeros of f_j cause problems. Thus, we choose Q such that f_j stays away from the zeros $(1 \le j \le d)$.
Let $a_{1,j} < \cdots < a_{k_j,j}$ be the zeros of f_j and define $a_{0,j} := 0$ and $a_{k_j+1,j} = 1$ (for $1 \le j \le d$). We then set for $\delta > 0$

$$Q := \bigcup_{\overline{q}\in\mathbb{N}^d} Q_{\overline{q}} \;\text{ with }\; Q_{\overline{q}} := \prod_{j=1}^{d} \left[a_{q_j,j} + \delta, a_{q_j+1,j} - \delta\right] \;\text{ and }\; \widetilde{Q}_{\overline{q}} := \prod_{j=1}^{d} \left[a_{q_j,j}, a_{q_j+1,j}\right]$$

Note that $\overline{q} = (q_1,\ldots,q_d) \in [0,\ldots,k_1+1] \times [0,\ldots,k_2+1] \times \cdots \times [0,\ldots,k_d+1]$ and we consider $Q_{\overline{q}}$ as empty if we have $q_j > k_j + 1$ for any j. An illustration of possible Q is given in Figure 3.1.

We will now adjust the Definitions 3.2.9, 3.2.10 and 3.2.11. The modification of Definition 3.2.9 is obvious. One simple takes h to be of bounded variation in the sense of Hardy and Krause in each $Q_{\overline{q}}$. The modification of Definition 3.2.10 is also straight forward. We call a face F of Q positive if there exists a $\overline{q} \in \mathbb{N}^d$ and a sequence $j_1, \cdots, j_k$ in $\{1, \ldots, d\}$ such that, for s_j $(1 \leq j \leq d)$ being the canonical coordinates in $[0,1]^d$,

$$F = \bigcap_{m=1}^{k} \left(\left\{ s_{j_m} = a_{q_{j_m}, j_m} - \delta \right\} \cap Q_{\overline{q}} \right).$$

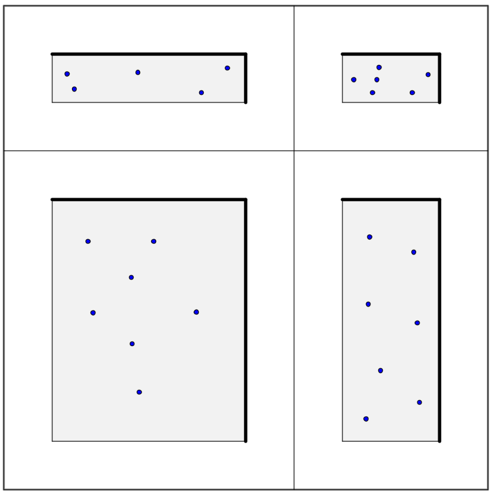

Figure 3.1: Illustration of Q, positive faces are bold

The modification of Definition 3.2.11 is slightly more tricky. Let F be a face of some $Q_{\overline{q}}$. Let $\boldsymbol{\varphi} \cap Q_{\overline{q}}$ be the subsequence of $\boldsymbol{\varphi}$ contained in $Q_{\overline{q}}$ and $\pi_F(\boldsymbol{\varphi} \cap Q_{\overline{q}})$ be the projection of $\boldsymbol{\varphi} \cap Q_{\overline{q}}$ to the face F. Unfortunately we cannot directly compute the discrepancy in the face F. We will see in the proof of Theorem 3.2.13 that we have to "extend F to the boundary of $\widetilde{Q}_{\overline{q}}$". More precisely this means to following: We set $\widetilde{F} := L \cap \widetilde{Q}_{\overline{q}}$, where L is the linear subspace containing F s.t. $\dim(L) = \dim(F)$ (see Figure 3.2 for an illustration). The discrepancy $D_n^*(F, \boldsymbol{\varphi})$ is then defined as the discrepancy of $\pi_F(\boldsymbol{\varphi} \cap Q_{\overline{q}})$ computed in $\widetilde{F}$.

We are now ready to state an extended version of Theorem 3.2.12.

Theorem 3.2.13. *Let $\delta > 0$ be fixed and $\boldsymbol{\varphi} = (\varphi^{(m)})_{m=1}^{n}$ be a sequence in Q. Let $h : Q \to \mathbb{C}$ be a function of bounded variation in the sense of Hardy and Krause.*

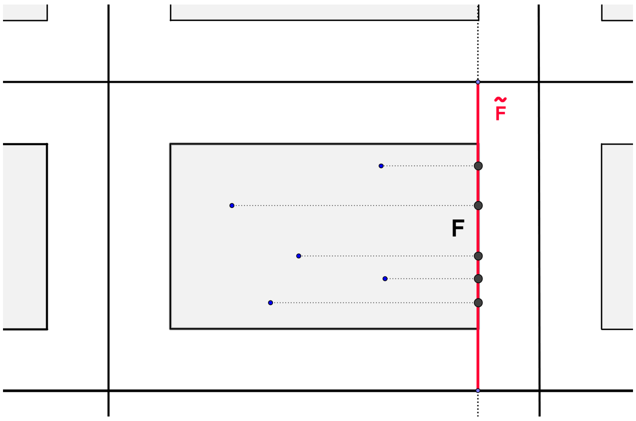

Figure 3.2: Illustration of $\widetilde{F}$

We then have

$$\left|\frac{1}{n}\sum_{m=1}^{n} h(\varphi^{(m)}) - \int_Q h(\overline{\phi})\ d\overline{\phi}\right| \leq \sum_{k=0}^{d-1} \delta^{d-k} \sum_{\substack{F \\ dim(F)=k}} \int_F h(\overline{\phi})\ dF(\overline{\phi}) \tag{3.2.16}$$

$$+\sum_{k=1}^{d} \sum_{\substack{F\ positive \\ dim(F)=k}} D_n^*(F,\boldsymbol{\varphi}) V(h|F).$$

Proof for $d=1$ and $d=2$. We assume that $Q=[\delta, 1-\delta]^d$. The more general case can be proven in the same way.
The idea is to modify the proof of Theorem 3.2.12 in [52]. There are indeed only minor modifications necessary. We present here only the cases $d=1$ and $d=2$ since we only need these two cases.
$\underline{d=1}$: We consider the integral $I_1 = I_1(h) := \int_\delta^{1-\delta} \big(A_n(\phi)-\phi\big)\ dh(\phi)$ with $A_n(\phi)$ given as in Definition 3.2.5.
It is clear from the definition of $D_n^*(\boldsymbol{\varphi})$ that

$$\left|\int_\delta^{1-\delta} \big(A_n(\phi)-\phi\big)\ dh(\phi)\right| \leq D_n^*(\boldsymbol{\varphi}) \int_\delta^{1-\delta} |dh(\phi)| = D_n^*(\boldsymbol{\varphi}) V(h|[\delta,1-\delta]). \tag{3.2.17}$$

On the other hand, one can use partial integration and partial summation to show that

$$I_1 = \big(\delta h(1-\delta)+\delta h(\delta)\big) + \int_\delta^{1-\delta} h(\phi)\ d\phi - \frac{1}{n}\sum_{m=1}^{n} h(\varphi^{(m)}). \tag{3.2.18}$$

This proves the theorem for $d=1$.
$\underline{d=2}$: In this case we consider the integral $I_2 = \int_{[\delta,1-\delta]^2} \big(A_n(\phi_1,\phi_2) - \phi_1\phi_2\big)\ dh(\phi_1,\phi_2)$. The argumentation is similar to the case $d=1$. As above, it

is immediate that I_2 is bounded by $D_n^*(\boldsymbol{\varphi})V(h|[\delta,1-\delta]^2)$. On the other hand, we get after consecutive partial integration

$$\begin{aligned}
&\int_{[\delta,1-\delta]^2} \phi_1\phi_2 \; dh(\phi_1,\phi_2) \\
&\quad = \sum_{k=0}^{2} \sum_{\substack{F \\ \dim(F)=k}} \delta^{2-k} \int_F h \; dF - \sum_{\substack{F \text{ positive} \\ \dim(F)=1}} \int_F h \; dF \\
&\qquad + h(1-\delta,1-\delta) - 2\delta h(1-\delta,1-\delta) - \delta h(\delta,1-\delta) - \delta h(1-\delta,\delta) \qquad (3.2.19)
\end{aligned}$$

and with two times partial summation

$$\begin{aligned}
&\int_{[\delta,1-\delta]^2} A_n(\phi_1,\phi_2) \; dh(\phi_1,\phi_2) \\
&\quad = h(1-\delta,1-\delta) - \sum_{\substack{F \\ \dim(F)=1}} \frac{1}{n}\sum_{m=1}^{n} h\big(\pi_F(\varphi^{(m)})\big) + \frac{1}{n}\sum_{m=1}^{n} h(\varphi^{(m)}). \qquad (3.2.20)
\end{aligned}$$

We now subtract (3.2.19) from (3.2.20) and expand the sum over the positive faces (with $\varphi^{(m)} = (\varphi_1^{(m)},\varphi_2^{(m)})$). We get

$$\begin{aligned}
I_2 = &\left(\frac{1}{n}\sum_{m=1}^{n} h(\varphi^{(m)}) - \int_Q h(\overline{\phi}) \; d\overline{\phi}\right) - \sum_{k=1}^{2} \sum_{\substack{F \\ \dim(F)=k}} \delta^{2-k} \int_F h \; dF \qquad (3.2.21) \\
&+ \left(\int_\delta^{1-\delta} h(u,1-\delta) \; du - \frac{1}{n}\sum_{m=1}^{n} h(\varphi_1^{(m)},1-\delta) + \delta h(\delta,1-\delta) + \delta h(1-\delta,1-\delta)\right) \qquad (3.2.22) \\
&+ \left(\int_\delta^{1-\delta} h(1-\delta,v) \; dv - \frac{1}{n}\sum_{m=1}^{n} h(1-\delta,\varphi_2^{(m)}) + \delta h(1-\delta,\delta) + \delta h(1-\delta,1-\delta)\right). \qquad (3.2.23)
\end{aligned}$$

The brackets (3.2.22) and (3.2.23) agree with (3.2.18) if we set "$h(s) = h(1-\delta,s)$" in (3.2.22), respectively "$h(s) = h(s,1-\delta)$" in (3.2.23). We thus can interpret the brackets (3.2.22) and (3.2.23) as integrals over the positive faces of Q and apply the induction hypothesis ($d=1$). A simple application of the triangle inequality proves the theorem for $d=2$.

It is important to point out that the discrepancy of $(\varphi_1^{(m)})_{m=1}^n$ and $(\varphi_2^{(m)})_{m=1}^n$ is computed in $[0,1]$ and not in $[\delta,1-\delta]$. This observation is the origin for the definition of $D_n^*(F,\boldsymbol{\varphi})$ before Theorem 3.2.13.

□

In Section 3.4.2, we will consider sums of the form

$$\frac{1}{n}\sum_{m=1}^{n} \log\left(f_j\left(e^{2\pi i m\varphi_j}\right)\right)\log\left(f_\ell\left(e^{2\pi i m\varphi_\ell}\right)\right). \qquad (3.2.24)$$

We are thus primary interested in (d-dimensional) sequences $\boldsymbol{\varphi}_{\mathrm{Kro}} = (\varphi_{\mathrm{Kro}}^{(m)})_{m=1}^{\infty}$, for given $\overline{\varphi} = (\varphi_1, \cdots, \varphi_d) \in \mathbb{R}^d$, defined as follows:

$$\varphi_{\mathrm{Kro}}^{(m)} = (\{m\varphi_1\}, \ldots, \{m\varphi_d\}), \tag{3.2.25}$$

where $\{s\} := s - [s]$ and $[s] := \max\{n \in \mathbb{Z}, n \leq s\}$. The sequence $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\mathrm{Kro}}$ is called *Kronecker-sequence of* $\overline{\varphi}$. The next lemma shows that the Kronecker-sequence is for almost all $\overline{\varphi} \in \mathbb{R}^d$ uniformly distributed.

Lemma 3.2.14. *Let $\overline{\varphi} = (\varphi_1, \ldots, \varphi_d) \in \mathbb{R}^d$ be given. The Kronecker-sequence of $\boldsymbol{\varphi}$ is uniformly distributed in $[0,1]^d$ if and only if $1, \varphi_1, \ldots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*

Proof. See [26, Theorem 1.76] □

Our aim is to apply Theorem 3.2.12 and Theorem 3.2.13 for Kronecker sequences. We thus have to estimate the discrepancy in this case and find a suitable $\delta > 0$. We start by giving an upper bound for the discrepancy.

Lemma 3.2.15. *Let $\overline{\varphi} = (\varphi_1, \ldots, \varphi_d) \in [0,1]^d$ be given with $1, \varphi_1, \ldots, \varphi_d$ linearly independent over $\mathbb{Z}$. Let $\boldsymbol{\varphi}$ be the Kronecker sequence of $\overline{\varphi}$. We then have for each $H \in \mathbb{N}$*

$$D_n^*(\boldsymbol{\varphi}) \leq 3^d \left(\frac{2}{H+1} + \frac{1}{n} \sum_{0 < \|\overline{q}\|_\infty \leq H} \frac{1}{r(\overline{q}) \|\overline{q} \cdot \overline{\varphi}\|} \right) \tag{3.2.26}$$

with $\|.\|_\infty$ being the maximum norm, $\|a\| := \inf_{n \in \mathbb{Z}} |a - n|$ and $r(\overline{q}) = \prod_{i=1}^d \max\{1, q_i\}$ for $\overline{q} = (q_1, \cdots, q_d) \in \mathbb{N}^d$.

Proof. The proof is a direct application of the Erdös-Turán-Koksma inequality (see [26, Theorem 1.21]). □

It is clear that we can use Lemma 3.2.15 to give an upper bound for the discrepancy, if we can find a lower bound for $\|\overline{q} \cdot \overline{\varphi}\|$. The most natural is thus to assume that $\overline{\varphi}$ fulfills some diophantine equation. In order to state this more precise, we give the following definition:

Definition 3.2.16. *Let $\overline{\varphi} \in [0,1]^d$ be given. We call $\overline{\varphi}$ of finite type if there exist constants $K > 0$ and $\gamma \geq 1$ such that*

$$\|\overline{q} \cdot \overline{\varphi}\| \geq \frac{K}{(\|\overline{q}\|_\infty)^\gamma} \quad \textit{for all } \overline{q} \in \mathbb{Z}^d \setminus \{0\}. \tag{3.2.27}$$

If $\overline{\varphi} = (\varphi_1, \ldots, \varphi_d)$ is of finite type, then it follows immediately from the definition that each φ_j is also of finite type and the sequence $1, \varphi_1, \ldots, \varphi_d$ is linearly independent over $\mathbb{Z}$.
One can now show the following:

Theorem 3.2.17. *Let $\overline{\varphi} \in [0,1]^d$ be of finite type and $\boldsymbol{\varphi}$ be the Kronecker sequence of $\overline{\varphi}$. Then*

$$D_n^*(\boldsymbol{\varphi}) = O(n^{-\alpha}) \quad \text{for some } \alpha > 0. \tag{3.2.28}$$

Proof. This theorem is a direct consequence of Lemma 3.2.15 and a simple computation. Further details can be found in [26, Theorem 1.80] or in [84]. □

As already mentioned above, we will consider in Section 3.4.2 sums of the form

$$\frac{1}{n}\sum_{m=1}^{n} \log\left(f_j\left(e^{2\pi i m\varphi_j}\right)\right)\log\left(f_\ell\left(e^{2\pi i m\varphi_\ell}\right)\right). \tag{3.2.29}$$

Surprisingly, it is not necessary to consider summands with more than two factors, even when we study the joint behavior at more than two points. We thus give the following definition:

Definition 3.2.18. *Let $x_1 = e^{2\pi i\varphi_1}, \ldots, x_d = e^{2\pi i\varphi_d}$ be given. We call both sequences $(x_j)_{j=1}^d$ and $(\varphi_j)_{j=1}^d$ pairwise of finite type, if we have for all $j \neq \ell$ that $(\varphi_j, \varphi_\ell) \in [0,1]^2$ is of finite type in the sense of Definition 3.2.16.*

3.3 Central Limit Theorems for the Symmetric Group

In this section, we state general Central Limit Theorems (CLT's) on the symmetric group. These theorems will allow us to prove CLT's for the logarithm of the characteristic polynomial and for multiplicative class functions.

3.3.1 One dimensional CLT

For a permutation $\sigma \in S_n$, chosen with respect to the Ewens distribution with parameter θ, let C_m be the random variable corresponding to the number of cycles of length m of σ. In order to state the CLT's on the symmetric group, we introduce random variables

$$A_n := \sum_{m=1}^{n}\sum_{k=1}^{C_m} X_{m,k}, \tag{3.3.1}$$

where we consider $X_{m,k}$ to be independent real valued random variables with $X_{m,k} \stackrel{d}{=} X_{m,1}$, for all $1 \leq m \leq n$ and $k \geq 1$. Furthermore, all $X_{m,k}$ are independent of σ. Of course, if $X_{m,k} = \mathrm{Re}(\log(1 - x^{-m}T_{m,k}))$ (or $\mathrm{Im}(\log(1 - x^{-m}T_{m,k}))$), then A_n is equal in law to the real (or imaginary) part of $\log Z_{n,z}(x)$, which is the logarithm of the characteristic polynomial of $M_{\sigma,z}$. This will be treated in Section 3.4.

We state the first result:

Theorem 3.3.1. *Let $\theta > 0$ be fixed. Assume that the sequence $X_{m,1}$ fulfills the following conditions*

(i) $\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1}|\right] = O(1)$,

(ii) $\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[X_{m,1}^2\right] \to V$, *as* $n \to \infty$

(iii) $\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1}|^3\right] = o\left((\log n)^{1/2}\right)$,

(iv) $\mathbb{E}\left[|X_{m,1}|\right] = O(\log m)$, *and*

(v) There exists a $p > 1/\theta$ *such that* $\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1}|^p\right] = O(1)$.

Then

$$\frac{A_n - \mathbb{E}\left[A_n\right]}{\sqrt{\log n}} \tag{3.3.2}$$

converges in law to the normal distribution $\mathcal{N}(0, \theta V)$.

Proof. For the proof, we will make use of the Feller coupling (see Section 3.2.1). This ensures that the random variables C_m and Y_m are defined on the same space and we can compare them with Lemma 3.2.3. The strategy of the proof is the following: We define

$$B_n = \sum_{m=1}^{n}\sum_{k=1}^{Y_m} X_{m,k}, \tag{3.3.3}$$

and show that A_n and B_n have the same asymptotic behavior after normalization. In particular, we will show the following lemma:

Lemma 3.3.2. *Suppose that the conditions of Theorem 3.3.1 hold. Then*

$$\mathbb{E}\left[|A_n - B_n|\right] = O(1). \tag{3.3.4}$$

In particular, it follows immediately that

$$\frac{A_n - B_n}{\sqrt{\log n}} \xrightarrow{d} 0 \quad \text{as } n \to \infty \tag{3.3.5}$$

and that the random variables $A_n - \mathbb{E}\left[A_n\right]$ *and* $B_n - \mathbb{E}\left[B_n\right]$ *have the same asymptotic behavior after normalization with* $\sqrt{\log n}$.

Proof of Lemma 3.3.2. We only have to prove (3.3.4). The other statements follow directly with Markov's inequality and Slutsky's theorem.
We use that $X_{m,k}$ is independent of C_m and Y_m and that $\mathbb{E}\left[X_{m,k}\right] = \mathbb{E}\left[X_{m,1}\right]$. We get

$$\begin{aligned}
&\mathbb{E}\left[|A_n - B_n|\right] \\
&= \mathbb{E}\left[\left|\sum_{m=1}^{n}\left(\sum_{k=1}^{C_m} X_{m,k} - \sum_{k=1}^{Y_m} X_{m,k}\right)\right|\right] \le \sum_{m=1}^{n}\mathbb{E}\left[\left|\sum_{k=(C_m\wedge Y_m)+1}^{C_m\vee Y_m} X_{m,k}\right|\right] \\
&\le \sum_{m=1}^{n}\mathbb{E}\left[\sum_{k=(C_m\wedge Y_m)+1}^{C_m\vee Y_m}\mathbb{E}\left[|X_{m,k}|\right]\right] \le \sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1}|\right]\mathbb{E}\left[|C_m - Y_m|\right]
\end{aligned} \tag{3.3.6}$$

By Lemma 3.2.3, there exists for any $\theta > 0$ a constant $K(\theta)$, such that

$$\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,k}|\right] \mathbb{E}\left[|C_m - Y_m|\right] \leq \frac{K(\theta)}{n} \sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1}|\right] + \frac{\theta}{n} \sum_{m=1}^{n} \Psi_n(m) \mathbb{E}\left[|X_{m,1}|\right]. \tag{3.3.7}$$

By assumption (3.3.1), it is clear that the first term on the RHS in (3.3.7) is always $O(1)$. For $\theta \geq 1$, the second term on the RHS is also bounded, since $\Psi_n(m) \leq 1$. We will use condition (3.3.1) for $\theta < 1$ and Lemma 3.2.4. Then, by the Hölder inequality, the second term on the RHS is bounded for $\theta < 1$, since

$$\begin{aligned}
&\frac{1}{n} \sum_{m=1}^{n} \Psi_n(m) \mathbb{E}\left[|X_{m,1}|\right] \\
&\leq K_2 \frac{\mathbb{E}\left[|X_{n,1}|\right]}{n} n^{1-\theta} + \frac{K_1}{n} \sum_{m=1}^{n-1} \mathbb{E}\left[|X_{m,1}|\right] \left(1 - \frac{m}{n}\right)^{\theta-1} \\
&\leq \frac{O(\log n)}{n^{\theta}} + \left(\frac{K_1}{n} \sum_{m=1}^{n-1} \mathbb{E}\left[|X_{m,1}|^p\right]\right)^{1/p} \cdot \left(\sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right)^{q(\theta-1)}\right)^{1/q},
\end{aligned} \tag{3.3.8}$$

where the constants K_1 and K_2 are chosen as in Lemma 3.2.4 and p is such that condition (3.3.1) is satisfied. By a simple change of variable $m \to n - m$, the second factor in (3.3.8) is bounded. This proves the lemma. □

Lemma 3.3.2 shows that it is enough to consider B_n. We now complete the proof of Theorem 3.3.1 by computing the characteristic function $\chi_n(t)$ of $B_n - \mathbb{E}[B_n]$, normalized by $\sqrt{\log n}$. To simplify the notation, we define the constant

$$K := \prod_{m=1}^{n} \exp\left(-\frac{it}{\sqrt{\log n}} \frac{\theta}{m} \mathbb{E}\left[X_{m,1}\right]\right). \tag{3.3.9}$$

Then,

$$\begin{aligned}
\chi_n(t) &:= \mathbb{E}\left[\exp\left(it \frac{B_n - \mathbb{E}[B_n]}{\sqrt{\log n}}\right)\right] \qquad (3.3.10) \\
&= \mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}} \sum_{m=1}^{n} \left(\sum_{k=1}^{Y_m} X_{m,k} - \frac{\theta}{m} \mathbb{E}\left[X_{m,k}\right]\right)\right)\right] \\
&= K \mathbb{E}\left[\prod_{m=1}^{n} \prod_{k=1}^{Y_m} \mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}} X_{m,k}\right)\right]\right] \\
&= K \prod_{m=1}^{n} \mathbb{E}\left[\left(\mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}} X_{m,1}\right)\right]\right)^{Y_m}\right] \\
&= K \prod_{m=1}^{n} \exp\left(\frac{\theta}{m}\left(\mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}} X_{m,1}\right)\right] - 1\right)\right) \\
&= K \prod_{m=1}^{n} \exp\left(\frac{\theta}{m} \mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}} X_{m,1}\right) - 1\right]\right)
\end{aligned}$$

We now use the fact that $|(e^{is}-1)-(is-s^2)| \leq |s^3|$ and get

$$\exp\left(\frac{it}{\sqrt{\log n}}X_{m,1}\right)-1=\frac{it}{\sqrt{\log n}}X_{m,1}-\frac{t^2}{2\log n}X_{m,1}^2+O\left(\frac{t^3X_{m,1}^3}{(\log n)^{3/2}}\right). \quad (3.3.11)$$

We combine (3.3.10), (3.3.11) and the definition of K to get

$$\chi_n(t)=\exp\left(-\theta\frac{t^2}{2\log n}\sum_{m=1}^{n}\frac{\mathbb{E}\left[X_{m,1}^2\right]}{m}+O\left(\frac{t^3}{(\log n)^{3/2}}\sum_{m=1}^{n}\frac{\mathbb{E}\left[X_{m,1}^3\right]}{m}\right)\right), \quad (3.3.12)$$

which goes to $\exp\left(-\theta\frac{t^2}{2}\right)$ by the assumptions (3.3.1), (3.3.1) and the following well-known lemma:

Lemma 3.3.3. *Let $(a_k)_{k\geq 1}$ be a real sequence. Suppose that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}a_k=L. \quad (3.3.13)$$

Then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{a_k}{k}=L. \quad (3.3.14)$$

This concludes the proof of Theorem 3.3.1 □

Remark 3.3.3.1. *From the proof of Theorem 3.3.1 it is clear, that the normalization is not restricted to $\sqrt{\log n}$. In fact, we could normalize by any term which goes to infinity with n. Also, it is worth to notice that condition 3.3.1 is optimal when it comes to regular permutation matrices. Then, the real part of the logarithm of the characteristic polynomial of $M_{\sigma,1}$ is determined by $X_{m,1}=\log|1-x^{-m}|=O(\log m)$. Condition 3.3.1 is only necessary for $\theta<1$ and can be omitted for $\theta\geq 1$.*

3.3.2 Multi dimensional central limit theorems

In this section, we replace the random variables $X_{m,k}$ in Theorem 3.3.1 by $\mathbb{R}^d$-valued random variables $\overline{X}_{m,k}=(X_{m,k,1},\dots,X_{m,k,d})$ and prove a CLT for

$$\overline{A}_{n,d}:=\sum_{m=1}^{n}\sum_{k=1}^{C_m}\overline{X}_{m,k}. \quad (3.3.15)$$

As before, we assume that $\overline{X}_{m,k}$ is a sequence of independent random variables such that $\overline{X}_{m,k}\overset{d}{=}\overline{X}_{m,1}$ and all $\overline{X}_{m,k}$ and $\sigma\in S_n$ are independent. We will prove the following theorem:

Theorem 3.3.4. *Let $\theta>0$ and $d\in\mathbb{N}$ be fixed. Furthermore, assume that for any $1\leq j\leq d$ the following conditions are satisfied:*

(i) $\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1,j}|\right]=O(1)$,

(ii) $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[X_{m,1,j}X_{m,1,\ell}\right] \to \sigma_{j,\ell}$, *as* $n \to \infty$,

(iii) $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[\left|X_{m,1,j}^3\right|\right] = o\left(\log^{1/2}(n)\right)$,

(iv) $\mathbb{E}\left[X_{m,1.j}\right] = O(\log m)$,

(v) *there exists a* $p > 1/\theta$ *s.t.* $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1,j}|^p\right] = O(1)$.

Then the distribution of

$$\frac{\overline{A}_{n,d} - \mathbb{E}\left[\overline{A}_{n,d}\right]}{\sqrt{\log n}} \tag{3.3.16}$$

converges in law to the normal distribution $\mathcal{N}(0, \theta\Sigma)$*, where* Σ *is the covariance matrix* $(\sigma_{ij})_{1\le i,j\le d}$.

Proof. The theorem follows from the Cramer-Wold theorem if we can show for each $\overline{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$

$$\overline{t} \cdot \frac{\overline{A}_n - \mathbb{E}\left[\overline{A}_n\right]}{\sqrt{\log n}} \xrightarrow{d} N(0, \overline{t}\Sigma\overline{t}^T). \tag{3.3.17}$$

A simple computation shows that

$$\overline{t} \cdot \overline{A}_n = \sum_{m=1}^{n}\sum_{k=1}^{C_m} V_{m,k}^{(d)} \tag{3.3.18}$$

with

$$V_{m,k}^{(d)} := V_{m,k} = \sum_{j=1}^{d} t_j X_{m,k,j}. \tag{3.3.19}$$

We now show that $V_{m,k}$ fulfills the conditions of Theorem 3.3.1. Clearly, $V_{m,k}$ is a sequence of independent random variables, $V_{m,k} \stackrel{d}{=} V_{m,1}$ and $V_{m,k}$ is independent of C_b for all m, k, b. The Conditions (3.3.1), (3.3.1) and (3.3.1) of Theorem 3.3.1 are straight forward. We thus proceed by verifying Conditions (3.3.1) and (3.3.1). We have

$$\begin{aligned} \frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[V_{m,1}^2\right] &= \frac{1}{n}\sum_{m=1}^{n}\sum_{j,\ell=1}^{d} t_j t_\ell \mathbb{E}\left[X_{m,1,j}X_{m,1,\ell}\right] \\ &\to \sum_{j,\ell=1}^{d} t_j t_\ell \sigma_{j,\ell} = \overline{t}\Sigma\overline{t}^T \end{aligned} \tag{3.3.20}$$

with $\Sigma = (\sigma_{j,\ell})_{1\le j,\ell\le d}$. This shows that (3.3.1) is fulfilled. We now look at (3.3.1). We use the generalized Hölder inequality and get

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[V_{m,1}^{3}\right] \le \frac{1}{n}\sum_{j_1,j_2,j_3=1}^{d}|t_{j_1}t_{j_2}t_{j_3}|\left(\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1,j_1}X_{m,1,j_2}X_{m,1,j_3}|\right]\right)$$
$$\le \sum_{j_1,j_2,j_3=1}^{d}\frac{|t_{j_1}t_{j_2}t_{j_3}|}{n}\left(\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1,j_1}|^3\right]^{1/3}\mathbb{E}\left[|X_{m,1,j_2}|^3\right]^{1/3}\mathbb{E}\left[|X_{m,1,j_3}|^3\right]^{1/3}\right)$$
$$\le \sum_{j_1,j_2,j_3=1}^{d}\frac{|t_{j_1}t_{j_2}t_{j_3}|}{n}\prod_{a=1}^{3}\left(\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1,j_a}|^3\right]\right)^{1/3} = o\left(\log^{1/2}(n)\right).$$

This concludes the proof of Theorem 3.3.4. □

Remark 3.3.4.1. *It is clear that Theorem 3.3.4 can be used for complex random variables, by identifying $\mathbb{C}$ by $\mathbb{R}^2$.*

3.4 Results on the Characteristic Polynomial and Multiplicative Class Functions

In this section we apply the theorems in Section 3.3 to the logarithm of the characteristic polynomial and the logarithm of multiplicative class functions. We study in Section 3.4.1 the of behavior of the real and imaginary part separately and then consider in Section 3.4.2 the joint behavior and the behavior at different points.
We first recall the branch of logarithm we use for $Z_{n,z}(x)$ and specify the branch of logarithm for $W_z^{1,n}(f)$ and $W_z^{2,n}(f)$, given by Definitions 3.1.2 and 3.1.3. As in Definition 3.1.1, it is natural to choose the branch of logarithm as follows:

Definition 3.4.1. *Let $x = e^{2\pi i\varphi} \in \mathbb{T}$ be a fixed number, z a $\mathbb{T}$–valued random variable and $f : \mathbb{T} \to \mathbb{C}$ a real analytic function. Furthermore, let $(z_{m,k})_{m,k=1}^{\infty}$ and $(T_{m,k})_{m,k=1}^{\infty}$ be two sequences of independent random variables, independent of $\sigma \in S_n$ with*

$$z_{m,k} \overset{d}{=} z \quad \text{and} \quad T_{m,k} \overset{d}{=} \prod_{j=1}^{m} z_{j,k}. \tag{3.4.1}$$

We then set

$$\log\bigl(Z_{n,z}(x)\bigr) := \sum_{m=1}^{n}\sum_{k=1}^{C_m}\log(1 - x^{-m}T_{m,k}), \tag{3.4.2}$$

$$w^{1,n}(f)(x) := \log\left(W_z^{1,n}(f)(x)\right) := \sum_{m=1}^{n}\sum_{k=1}^{C_m}\log\bigl(f(x^m z_{m,k})\bigr), \tag{3.4.3}$$

$$w^{2,n}(f) := \log\left(W_z^{2,n}(f)(x)\right) := \sum_{m=1}^{n}\sum_{k=1}^{C_m}\log\bigl(f(x^m T_{m,k})\bigr), \tag{3.4.4}$$

where we have used for $\log(.)$ *the principal branch of logarithm. We will deal with negative values as follows:* $\log(-y) = \log y + i\pi$, $y \in \mathbb{R}_+$. *Note, that it is not necessary to specify the logarithm at* 0, *since we will deal only with cases where this occurs with probability* 0.

3.4.1 Limit behavior at 1 point

We will discuss some important cases for which the conditions in Theorem 3.3.1 are fulfilled. The results in this section are

Theorem 3.4.2. *Let* S_n *be endowed with the Ewens distribution with parameter* θ, f *be a non zero real analytic function,* z *a* $\mathbb{T}$*-valued random variable and* $x = e^{2\pi i\varphi} \in \mathbb{T}$ *be not a root of unity, i.e.* $x^m \neq 1$ *for all* $m \in \mathbb{Z}$.
Suppose that one of the following conditions is fulfilled,

- z *is uniformly distributed,*
- z *is absolutely continuous with bounded, Riemann integrable density,*
- z *is discrete, there exists a* $\rho > 0$ *with* $z^\rho \equiv 1$, *all zeros of* f *are roots of unity and* x *is of finite type (see Definition 3.2.16).*

We then have

$$\frac{\operatorname{Re}\left(w^{1,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_R(f)\sqrt{\log n} \xrightarrow{d} N_R, \tag{3.4.5}$$
$$\frac{\operatorname{Im}\left(w^{1,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_I(f)\sqrt{\log n} \xrightarrow{d} N_I \tag{3.4.6}$$

with $N_R \sim \mathcal{N}\left(0, \theta V_R(f)\right), N_I \sim \mathcal{N}\left(0, \theta V_I(f)\right)$ *and*

$$m_R(f) = \operatorname{Re}\left(\int_0^1 \log\bigl(f(e^{2\pi i\phi})\bigr)\, d\phi\right), \quad V_R(f) = \int_0^1 \log^2\bigl|f(e^{2\pi i\phi})\bigr|\, d\phi, \tag{3.4.7}$$
$$m_I(f) = \operatorname{Im}\left(\int_0^1 \log\bigl(f(e^{2\pi i\phi})\bigr)\, d\phi\right), \quad V_I(f) = \int_0^1 \arg^2\bigl(f(e^{2\pi i\phi})\bigr)\, d\phi. \tag{3.4.8}$$

Theorem 3.4.3. *Let* S_n *be endowed with the Ewens distribution with parameter* θ, f *be a non zero real analytic function,* z *a* $\mathbb{T}$*-valued random variable and* $x \in \mathbb{T}$ *be not a root of unity.*
Suppose that one of the following conditions is fulfilled,

- z *is uniformly distributed,*
- z *is absolutely continuous with density* $g : [0,1] \to \mathbb{R}_+$, *such that*

$$g(\phi) = \sum_{j\in\mathbb{Z}} c_j e^{2\pi i j\phi} \quad \text{with} \quad \sum_{j\in\mathbb{Z}} |c_j| < \infty. \tag{3.4.9}$$

- *z is discrete, there exists a $\rho > 0$ with $z^\rho \equiv 1$, all zeros of f are roots of unity, x is of finite type (see Definition 3.2.16) and for each $1 \leq k \leq \rho$,*

$$\mathbb{P}\left[z = e^{2\pi ik/\rho}\right] = \frac{1}{\rho}\sum_{j=0}^{\rho-1} c_j e^{2\pi ijk} \quad \textit{with } |c_j| < 1 \textit{ for } j \neq 0. \tag{3.4.10}$$

We then have

$$\frac{\mathrm{Re}\left(w^{2,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_R(f)\sqrt{\log n} \xrightarrow{d} N_R, \tag{3.4.11}$$

$$\frac{\mathrm{Im}\left(w^{2,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_I(f)\sqrt{\log n} \xrightarrow{d} N_I, \tag{3.4.12}$$

with $m_R(f), m_I(f), N_R$ and N_I as in Theorem 3.4.2.

Since $Z_{n,z}(x)$ is the special case $f(x) = 1 - x^{-1}$ of W^2, we get immediately with a short computation the following corollary, which covers Proposition 3.1.1:

Corollary 3.4.3.1. *Let S_n be endowed with the Ewens distribution with parameter θ, z a $\mathbb{T}$-valued random variable and $x \in \mathbb{T}$ be not a root of unity, i.e. $x^m \neq 1$ for all $m \in \mathbb{Z}$.*
Suppose that one of the conditions in Theorem 3.4.3 holds, then

$$\frac{\mathrm{Re}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\log n}} \xrightarrow{d} N_R \quad \textit{and} \tag{3.4.13}$$

$$\frac{\mathrm{Im}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\log n}} \xrightarrow{d} N_I, \tag{3.4.14}$$

with $N_R, N_I \sim \mathcal{N}\left(0, \theta\frac{\pi^2}{12}\right)$.

In Corollary 3.4.3.1 $\mathrm{Re}\left(\log\left(Z_{n,z}(x)\right)\right)$ and $\mathrm{Im}\left(\log\left(Z_{n,z}(x)\right)\right)$ are converging to normal random variables without centering. This is due to that the expectation is $o(\sqrt{\log n})$. This will become more clear below.

Remark 3.4.3.1. *The case x a root of unity can be treated similarly. The computations are indeed much simpler, see for instance [83] for $z \equiv 1$.*

The rest of this section is devoted to the proofs of Theorem 3.4.2 and 3.4.3. We have to distinguish the cases where z is absolutely continuous and z is discrete. Thus, we divide the proof into subsections. We will verify the assumptions of Theorem 3.3.1 only for the real part, since the computations for the imaginary part are much simpler and easy to show. To illustrate the computations, we begin with the simplest case.

Proofs of Theorem 3.4.2 and 3.4.3:

3.4.1.1 Uniform measure on the unit circle

We consider z to be uniformly distributed on the unit circle $\mathbb{T}$. Under this condition, we prove that Theorem 3.4.2, Theorem 3.4.3 and Corollary 3.4.3.1 hold. We start with the proof of Corollary 3.4.3.1.

Proof Corollary 3.4.3.1 for uniform z. We start with the characteristic polynomial. We set

$$X_{m,k} := \mathrm{Re}\big(\log(1 - x^{-m}T_{m,k})\big) = \log|1 - x^{-m}T_{m,k}|. \tag{3.4.15}$$

It is easy to check that in this case $T_{m,k}$ is also uniformly distributed. Thus,

$$x^{-m}T_{m,k} \stackrel{d}{=} T_{m,k} \stackrel{d}{=} z \stackrel{d}{=} z_{m,k} \quad \text{and} \quad X_{m,k} \stackrel{d}{=} \log|1 - z_{m,k}|. \tag{3.4.16}$$

We have

$$\mathbb{E}\left(|X_{m,k}|\right) = \int_0^1 \left|\log|1 - e^{-2i\pi\phi}|\right| \, d\phi = \int_{-1/2}^{1/2} \left|\log|1 - e^{-2i\pi\phi}|\right| \, d\phi \tag{3.4.17}$$

This integral exists, since $|1 - e^{-2i\pi\phi}| \sim 2\pi\phi$ for $\phi \to 0$, in particular,

$$\mathbb{E}\left[|X_{m,1}|\right] \leq \int_{-1/2}^{1/2} \left|\log\left|\frac{1 - e^{-2i\pi\phi}}{\phi}\right|\right| \, d\phi + \int_{-1/2}^{1/2} |\log|\phi|| \, dt < \infty. \tag{3.4.18}$$

This shows that the first moment exists and so naturally, condition (3.3.1) and (3.3.1) of Theorem 3.3.1 are fulfilled by the independency of m and the upper bound in (3.4.18). We proceed by showing that the conditions (3.3.1), (3.3.1) and (3.3.1) hold for uniformly chosen z.
One can use partial integration and induction to see that $\mathbb{E}\left[|X_{m,1}|^p\right]$ exists for all $p \geq 1$. Moreover, as for $p = 1$, $\mathbb{E}\left[|X_{m,1}|^p\right]$ is independent of m. In particular,

$$\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[\log^2|1 - z_{m,1}|\right] = \int_0^1 \log^2|1 - e^{-2i\pi\phi}| \, d\phi = \frac{\pi^2}{12}. \tag{3.4.19}$$

We thus have

$$\frac{\mathrm{Re}\left(\log(Z_{n,z})\right) - \mathbb{E}\left[\mathrm{Re}\left(\log(Z_{n,z})\right)\right]}{\sqrt{\log n}} \xrightarrow{d} N_R \tag{3.4.20}$$

with $N_R \sim \mathcal{N}\left(0, \theta\frac{\pi^2}{12}\right)$.
It remains to show that $\mathbb{E}\left[\mathrm{Re}\left(\log(Z_{n,z}(x))\right)\right]$ is $o(\sqrt{\log n})$. We have

$$\begin{aligned}\mathbb{E}\left[\mathrm{Re}\left(\log(Z_{n,z}(x))\right)\right] &= \sum_{m=1}^{n} \mathbb{E}\left[C_m\right]\mathbb{E}\left[\log|1 - z_{m,1}|\right] \\ &= \left(\int_0^1 \log|1 - e^{-2i\pi\phi}| \, d\phi\right)\left(\sum_{m=1}^{n} \mathbb{E}\left[C_m\right]\right). \end{aligned}\tag{3.4.21}$$

By Jensen's formula, $\int_0^1 \log|1-e^{-2i\pi\phi}|\, d\phi = 0$ []. This completes the proof for the real part of $Z_{n,z}(x)$.

For the imaginary part we use that $\mathrm{Im}\big(\log(1-e^{-2\pi i\phi})\big) = \frac{\pi}{2} - \phi\pi$ for $\phi \in [0,1[$. By similar computations as for the real part, it is easy to see that all the conditions of Theorem 3.3.1 are fulfilled and thus, Corollary 3.4.3.1 holds for uniform z. □

We now proceed with multiplicative class functions, for uniform z. We start by giving the proof of Theorem 3.4.2.

Proof of Theorem 3.4.2 and Theorem 3.4.3 for uniform z. Since $T_{m,k}$ is uniformly distributed, we have $T_{m,k} \stackrel{d}{=} z_{m,k}$ and thus $w^{1,n}(f) \stackrel{d}{=} w^{2,n}(f)$. We therefore do not have to distinguish these cases. Furthermore, if $x_0 = e^{2\pi i\phi_0}$ is a zero of f, then the real part behaves as follows:

$$\log\big|f(e^{2\pi i\phi})\big| \sim K \log|\phi-\phi_0|, \tag{3.4.22}$$

for $\phi \to \phi_0$ and a $K > 0$. Also, the imaginary part $\arg\big(f(e^{2\pi i\phi})\big)$ is bounded and piecewise real analytic with at most finitely many discontinuity points. This shows that all expectations exist and we can use the same argumentation as for $\log\big(Z_{n,z}(x)\big)$ in the proof of Corollary 3.4.3.1, for z being uniformly distributed.
The verification of the assumptions of Theorem 3.3.1 are then straight forward and we thus omit the details.
The only point that needs a little bit more explanation is the behavior of $\mathbb{E}\left[\mathrm{Re}\left(w^{j,n}(f)\right)\right]$ and $\mathbb{E}\left[\mathrm{Im}\left(w^{j,n}(f)\right)\right]$. We can use Lemma 3.3.2 and get

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Re}\left(\log\big(Z_{n,z}(x)\big)\right)\right] &= \sum_{m=1}^{n} \mathbb{E}\left[C_m\right]\mathbb{E}\left[\log\big|f(z_{m,1})\big|\right] \\
&= \left(\sum_{m=1}^{n} \mathbb{E}\left[Y_m\right]\mathbb{E}\left[\log\big|f(z_{m,1})\big|\right]\right) + O(1) \\
&= \left(\int_0^1 \log\big|f(e^{2\pi i\phi})\big|\, d\phi\right)\left(\sum_{m=1}^{n}\frac{\theta}{m}\right) + O(1) \\
&= \theta \cdot m_R(f)\log n + O(1).
\end{aligned} \tag{3.4.23}$$

This works similar for the imaginary part. So, this completes the proof of the case where z is uniformly distributed. □

3.4.1.2 Absolute continuous on the unit circle

We consider here z absolutely continuous. We assume that the density g of z is bounded and Riemann integrable, i.e

$$\mathbb{P}\left[z \in [e^{2\pi i\alpha}, e^{2\pi i\beta}]\right] = \int_\alpha^\beta g(\phi)\, d\phi \quad \text{for } 0 \le \alpha \le \beta \le 1. \tag{3.4.24}$$

In this situation, $w^{1,n}(f) \neq w^{2,n}(f)$ and we thus have to distinguish this cases. In the following, we only give the proofs for Theorem 3.4.2 and 3.4.3, since Corollary 3.4.3.1 follows immediately from Theorem 3.4.3.
We begin with $w^{1,n}(f)$ since the computations are simpler.

Proof of Theorem 3.4.2 for absolutely continuous z. We set

$$X_{m,k} := \log\left|f(z_{m,k}x^m)\right| \tag{3.4.25}$$

and $x = e^{2\pi i\varphi}$. For simplicity, we write $h(\phi) := \log\left|f(e^{2\pi i\phi})\right|$. We now show that the assumptions of Theorem 3.3.1 are fulfilled. It is easy to see that

$$\begin{aligned}\mathbb{E}\left[|X_{m,k}|\right] &= \int_0^1 \left|\log\left|f(x^m e^{2\pi i\phi})\right|\right| g(\phi)\ d\phi = \int_0^1 |h(\phi+m\varphi)| g(\phi)\ d\phi \\ &\leq \sup_{\alpha\in[0,1]} |g(\alpha)| \int_0^1 |h(\phi)|\ d\phi < \infty. \end{aligned}\tag{3.4.26}$$

This shows that the first moment can be bounded independently of m and so assumptions (3.3.1) and (3.3.1) are fulfilled.

We now verify the other assumptions of Theorem 3.3.1. For simplicity, we will consider g as a periodic function with periodicity 1. We then have for any $p \geq 1$

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1}|^p\right] &= \frac{1}{n}\sum_{m=1}^{n}\int_0^1 |h(m\varphi+\phi)|^p g(\phi)\ d\phi = \frac{1}{n}\sum_{m=1}^{n}\int_0^1 |h(\phi)|^p g(\phi-m\varphi)\ d\phi \\ &= \int_0^1 |h|(\phi)|^p \left(\frac{1}{n}\sum_{m=1}^{n} g(\phi-m\varphi)\right)\ d\phi. \end{aligned}\tag{3.4.27}$$

All integrals in (3.4.27) exist, since g is bounded. We now take a closer look at $\frac{1}{n}\sum_{m=1}^{n} g(\phi-m\varphi)$. By assumption, $x = e^{2i\pi\varphi}$ is not a root of unity and φ is thus irrational. Therefore, $(\{m\varphi\})_{m=1}^{\infty}$ is uniformly distributed in $[0,1]$ and we can apply Theorem 3.2.6 for fixed ϕ, since g is Riemann integrable. We get as $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n} g(\phi-m\varphi) = \frac{1}{n}\sum_{m=1}^{n} g(\phi-\{m\varphi\}) \longrightarrow \int_0^1 g(\phi)\ d\phi. \tag{3.4.28}$$

We now can use dominated convergence in (3.4.27), since g is bounded and h^p is integrable. We get

$$\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,k}|^p\right] \to \int_0^1 |h(\phi)|^p\ d\phi. \tag{3.4.29}$$

As above, the arguments can be also applied for $X_{m,k} = \arg(f(z_{m,k}x^m))$. So, all assumptions of Theorem 3.3.1 are satisfied for both, the real and the imaginary part of $w^{1,n}(f)$.
It remains to show that the real part of $\mathbb{E}\left[w^{1,n}(f)\right]$ can be replaced by $\theta \cdot m_R(f)\log n$ and the imaginary part by $\theta \cdot m_I(f)\log n$. But this is clear by (3.4.29). So, this concludes the proof for absolutely continuous z. □

We now continue with $w^{2,n}$.

Proof of Theorem 3.4.3 for absolutely continuous z. We set in this case

$$X_{m,k} := \log\big|f(x^m T_{m,k})\big|. \tag{3.4.30}$$

The density of $T_{m,k}$ is g^{*m}, where g^{*m} is the $m-$times convolution of g with itself. We have by assumption

$$g(\phi) = \sum_{j\in\mathbb{Z}} c_j e^{2\pi i j\phi} \ \text{ with } \ |c_j| \le 1 \text{ for } j \neq 0 \ \text{ and } \ \sum_{j\in\mathbb{Z}} |c_j| < \infty. \tag{3.4.31}$$

We use that $\widehat{g^{*m}}(j) = (c_j)^m$ and write as above $h(\phi) := \log\big|f(e^{2\pi i\phi})\big|$. We get

$$\begin{aligned}\mathbb{E}\left[|X_{m,k}|\right] &= \int_0^1 |h(\phi + m\varphi)| g^{*m}(\phi)\ d\phi \le \sum_{j\in\mathbb{Z}} |c_j|^m \int_0^1 |h(\phi)|\ d\phi \\ &\le \sum_{j\in\mathbb{Z}} |c_j| \int_0^1 |h(\phi)|\ d\phi < \infty \end{aligned} \tag{3.4.32}$$

This shows that again, the first moment can be bounded independently of m and so, assumptions (3.3.1) and (3.3.1) of Theorem 3.3.1 are satisfied. We now come to assumptions (3.3.1), (3.3.1) and (3.3.1).
Let $p \ge 1$ be given, then

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^n \mathbb{E}\left[|X_{m,k}|^p\right] &= \frac{1}{n}\sum_{m=1}^n \int_0^1 |h(\phi + m\varphi)|^p g^{*m}(\phi)\ d\phi \\ &= \int_0^1 |h(\phi)|^p \left(\frac{1}{n}\sum_{m=1}^n g^{*m}(\phi - m\varphi)\right)\ d\phi. \end{aligned} \tag{3.4.33}$$

Consider now $\frac{1}{n}\sum_{m=1}^n g^{*m}(\phi - m\varphi)$. By assumption,

$$g(\phi) = \sum_{j\in\mathbb{Z}} c_j e^{2\pi i j\phi} \ \text{ with } \ |c_j| \le 1 \text{ for } j \neq 0 \ \text{ and } \ \sum_{j\in\mathbb{Z}} |c_j| < \infty. \tag{3.4.34}$$

We use that $\widehat{g^{*m}}(j) = (c_j)^m$ and get

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^n g^{*m}(\phi - m\varphi) &= \frac{1}{n}\sum_{m=1}^n \sum_{j\in\mathbb{Z}} c_j^m e^{2\pi i j(\phi - m\varphi)} \\ &= \sum_{j\in\mathbb{Z}} e^{2\pi i j\phi} \left(\frac{1}{n}\sum_{m=1}^n c_j^m e^{-2\pi i j m\varphi}\right). \end{aligned} \tag{3.4.35}$$

We now compute the behavior of $\frac{1}{n}\sum_{m=1}^n c_j^m e^{-2\pi i j m\varphi}$. For $j = 0$, this expression is just 1, since $c_0 = \int_0^1 g(\phi)\ d\phi = 1$. For $j \neq 0$, we use that $|c_j| \le 1$ and get for some constant K and almost all ϕ

$$\frac{1}{n}\left|\sum_{m=1}^n c_j^m e^{-2\pi i j m\varphi}\right| \le \frac{1}{n}\frac{1 - c_j^{n+1} e^{2i\pi j\phi(n+1)}}{1 - c_j e^{2i\pi j\phi}} \le \frac{K}{n}. \tag{3.4.36}$$

Also, we have $\frac{1}{n}\sum_{m=1}^{n}|c_j^m| \leq |c_j|$ and thus

$$\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi) \leq \left|\sum_{j\in\mathbb{Z}} e^{2\pi i j\phi}\left(\frac{1}{n}\sum_{m=1}^{n} c_j^m e^{-2\pi i j m\varphi}\right)\right| \leq \sum_{j\in\mathbb{Z}}|c_j| < \infty \quad (3.4.37)$$

So we can apply dominated convergence in (3.4.35). Therefore, as $n\to\infty$,

$$\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi) \longrightarrow 1 \quad (n\to\infty). \quad (3.4.38)$$

Furthermore, $\sum_j |c_j|$ is also an upper bound for $\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi)$. So again, we can use in (3.4.33) dominated convergence to get

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,k}|^p\right] \to \int_0^1 |h|^2(\phi)\, d\phi. \quad (3.4.39)$$

This completes the proof for absolutely continuous z.

□

3.4.1.3 Discrete measure on the unit circle

In this section, we will prove Theorem 3.4.2 and 3.4.3 for discrete z with $z^\rho \equiv 1$. The special case $z \equiv 1$ has been considered by Hambly, Keevash, O'Connell and Stark [40] for the (regular) characteristic polynomial and the uniform distribution, and by Zeindler [83] for multiplicative class functions and the Ewens distribution on S_n.
We need here the following result proven in [83, p. 14–15].

Lemma 3.4.4. *Let $f:\mathbb{T}\to\mathbb{C}$ be real analytic with only roots of unity as zeros and let $x = e^{2\pi i\varphi}$ be of finite type (see Definition 3.2.16). We then have $\log\bigl|f(x^n)\bigr| = O\bigl(\log n\bigr)$ and for any $p \geq 1$*

$$\frac{1}{n}\sum_{m=1}^{n}\log^p\bigl|f(x^m)\bigr| \longrightarrow \int_0^1 \log^p\bigl|f(e^{2\pi i\phi})\bigr|\, d\phi. \quad (3.4.40)$$

$$\frac{1}{n}\sum_{m=1}^{n}\bigl|\log^p\bigl|f(x^m)\bigr|\bigr| \longrightarrow \int_0^1 \Bigl|\log^p\bigl|f(e^{2\pi i\phi})\bigr|\Bigr|\, d\phi. \quad (3.4.41)$$

The idea of the proof of Lemma 3.4.4 is to use Theorem 3.2.13 for $d = 1$. The assumption x of finite type allows to choose a suitable δ and to estimate the discrepancy D_n^*.

Remark 3.4.4.1. *The upper bound for $\log\bigl|f(x^n)\bigr|$ in Lemma 3.4.4 is optimal, i.e. if f has a zero, then there exists a infinite sequence $(n_k)_{k=1}^{\infty}$ and a constant $K > 0$ such that $n_k \to \infty$ and*

$$\bigl|\log\bigl|f(x^{n_k})\bigr|\bigr| \geq K\log(n_k). \quad (3.4.42)$$

We begin as in Section 3.4.1.2 with the proof for $w^{1,n}(f)$.

Proof of Theorem 3.4.2 for discrete z. By $z^\rho \equiv 1$, we have for any $p \geq 1$

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[\log^p\left|f(z_{m,1}x^m)\right|\right] = \sum_{k=1}^{\rho}\mathbb{P}\left[z = e^{2\pi ik/\rho}\right]\left(\frac{1}{n}\sum_{m=1}^{n}\log^p\left|f(e^{2\pi ik/\rho}x^m)\right|\right). \tag{3.4.43}$$

We set $f_k(y) := f(e^{2\pi ik/\rho}y)$ for any $y \in \mathbb{T}$. By assumption, the zeros of f are only roots of unity and thus, the zeros of f_k are also only roots of unity. This shows that we can apply Lemma 3.4.4 in (3.4.43) for each k and we immediately get as $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[\log^p\left|f(z_{m,1}x^m)\right|\right] \to \int_0^1 \log^p\left|f(e^{2\pi i\phi})\right|\,d\phi. \tag{3.4.44}$$

So, the moments of $w^{1,n}(f)$ follow the same behavior as in the uniform case. By the proof of Theorem 3.4.2 for uniformly distributed z, we conclude the proof for discrete z.

□

We now give the proof for $w^{2,n}(f)$, if z is discrete.

Proof of Theorem 3.4.3 for discrete z. We start the proof by recalling, that for discrete z with $z^\rho \equiv 1$, there exist always a sequence $(c_j)_{0\leq j\leq \rho-1}$ such that

$$\mathbb{P}\left[z = e^{2\pi ik/\rho}\right] = \frac{1}{\rho}\sum_{j=0}^{\rho-1}c_j e^{2\pi ijk}. \tag{3.4.45}$$

(See for more details [69], chapter 7.)
It follows immediately

$$\mathbb{P}\left[T_{m,1} = e^{2\pi ik/\rho}\right] = \frac{1}{\rho}\sum_{j=0}^{\rho-1}c_j^m e^{2\pi ijk}. \tag{3.4.46}$$

For any $p \geq 1$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[\log^p\left|f(x^mT_{m,1})\right|\right] &= \frac{1}{n}\sum_{m=1}^{n}\sum_{k=0}^{\rho-1}\log^p\left|f(x^m e^{2\pi ik/\rho})\right|\mathbb{P}\left[T_{m,1} = e^{2\pi ijk/\rho}\right] \\
&= \frac{1}{n\rho}\sum_{m=1}^{n}\sum_{k=0}^{\rho-1}\sum_{j=0}^{\rho-1}c_j^m e^{2\pi ijk}\log^p\left|f(x^m e^{2\pi ik/\rho})\right| \\
&= \frac{1}{\rho}\sum_{k=0}^{\rho-1}\sum_{j=0}^{\rho-1}e^{2\pi ijk}\left(\frac{1}{n}\sum_{m=1}^{n}c_j^m\log^p\left|f(x^m e^{2\pi ik/\rho})\right|\right).
\end{aligned} \tag{3.4.47}$$

First, we consider only summands with $j \neq 0$ and we show that they vanish in the limit. Let k be fixed and $\epsilon > 0$ be arbitrary. Since by assumption $|c_j| < 1$, we can find a m_0 such that $|c_j|^m < \epsilon$ for $m \geq m_0$. We get

$$\left| \frac{1}{n} \sum_{m=1}^{n} c_j^m \log^p \left| f(x^m e^{2\pi i k/\rho}) \right| \right| \leq \frac{1}{n} \sum_{m=1}^{n} |c_j|^m \left| \log^p \left| f(x^m e^{2\pi i k/\rho}) \right| \right|$$
$$\leq o(1) + \frac{\epsilon}{n} \sum_{m=m_0}^{n} \left| \log^p \left| f(x^m e^{2\pi i k/\rho}) \right| \right| \quad (3.4.48)$$

Since x and f satisfy the assumptions of Lemma 3.4.4, we can use (3.4.41) to see that the last sum converges to $\epsilon \int_0^1 \log^p \left| f(e^{2\pi i \phi}) \right| \, d\phi$. This proves that the terms are vanishing in the limit, since ϵ was arbitrary. The remaining terms in (3.4.47), i.e. the terms with $j = 0$, are thus

$$\frac{1}{\rho} \sum_{k=0}^{\rho-1} \left(\frac{1}{n} \sum_{m=1}^{n} \log^p \left| f(x^m e^{2\pi i k/\rho}) \right| \right). \quad (3.4.49)$$

Again, by Lemma 3.4.4 we get as $n \to \infty$

$$\frac{1}{n} \sum_{m=1}^{n} \mathbb{E} \left[\log^p \left| f(x^m T_{m,k}) \right| \right] \longrightarrow \int_0^1 \log^p \left| f(e^{2\pi i \phi}) \right| \, d\phi. \quad (3.4.50)$$

So, the moments of $w^{2,n}(f)$ follow the same behavior as in the uniform case. By the proof of Theorem 3.4.3 for uniformly distributed z, we conclude the proof for discrete z.

□

This completes the proofs of Theorem 3.4.2 and 3.4.3.

3.4.2 Behavior at different points

In this section, we study the joint behavior of the real and the imaginary parts of the characteristic polynomial of $M_{\sigma,z}$ and of multiplicative class functions. Furthermore, we consider the behavior at a finite set of different points $x_1 = e^{2\pi i \varphi_1}, \ldots, x_d = e^{2\pi i \varphi_d}$, $d \in \mathbb{N}$ fixed. We will follow the structure of Section 3.4.1.

Before we state the results of this section, it is important to emphasize that we will allow different random variables $z_1, \ldots, z_d$ at the different points $x_1, \ldots, x_d$. Of course, we need to specify the joint behavior at the different points. For the multiplicative class function $w^{1,n}(f_j)(x_j)$, we define the following joint behavior. Let $\overline{z} = (z_1, \ldots, z_d)$ be a random variable with values in $\mathbb{T}^d$. Let further $\overline{z}^{(m,k)} = (z_1^{(m,k)}, \ldots, z_d^{(m,k)})$ be a sequence of i.i.d. random variables with $\overline{z}^{(m,k)} \stackrel{d}{=} \overline{z}$ (in m and k, for $1 \leq m \leq n$ and $1 \leq k \leq C_m$, where C_m denotes the number of cycles of m in σ). Then, for functions $f_1, \ldots, f_d$ and for any fixed $1 \leq j \leq d$,

$$w^{1,n}(f_j)(x_j) = w^{1,n}_{z_j}(f_j)(x_j) := \sum_{m=1}^{n}\sum_{k=1}^{C_m} \log\left(f_j\left(z_j^{(m,k)} x_j^m\right)\right). \tag{3.4.51}$$

This means that the behavior in different cycles of σ is independent. But the behavior in a given cycle at different points is determined by $\overline{z}$.

For the logarithm of the characteristic polynomial $\log\bigl(Z_{n,z}(x_j)\bigr)$ and for the multiplicative class function $w^{2,n}(f_j)(x_j)$, we do something similar. Intuitively, we construct for each point x_j a matrix M_{σ,z_j} as in (3.1.1), where we choose for M_{σ,z_1} n i.i.d. random variables, which are equal in distribution to z_1. At point x_2, we choose again n i.i.d random variables, which are equal in distribution to z_2 and so on. Formally, we define for (the same sequence as above) $\overline{z}^{(m,k)} = (z_1^{(m,k)}, \dots, z_d^{(m,k)})$ another sequence (in m and in k) $\overline{T}^{(m,k)} = (T_1^{(m,k)}, \dots, T_d^{(m,k)})$ of independent random variables, so that for any fixed $1 \le j \le d$ and fixed $1 \le m \le n$,

$$T_j^{(m,k)} \stackrel{d}{=} \prod_{\ell=1}^{m} z_j^{(m,\ell)}$$

and

$$(T_1^{(m,k)}, \dots, T_d^{(m,k)}) \stackrel{d}{=} \left(\prod_{\ell=1}^{m} z_1^{(m,\ell)}, \dots, \prod_{\ell=1}^{m} z_d^{(m,\ell)}\right). \tag{3.4.52}$$

This gives for fixed j's and function f_j:

$$w^{2,n}(f_j)(x_j) = w^{2,n}_{z_j}(f_j)(x_j) := \sum_{m=1}^{n}\sum_{k=1}^{C_m} \log\left(f_j\left(T_j^{(m,k)} x_j^m\right)\right). \tag{3.4.53}$$

We now state the results of this section:

Theorem 3.4.5. *Let S_n be endowed with the Ewens distribution with parameter θ, $f_1, \dots, f_d$ be non zero real analytic functions, $\overline{z} = (z_1, \dots, z_d)$ a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i\varphi_1}, \dots, x_d = e^{2\pi i\varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \dots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*
Suppose that one of the following conditions is satisfied:

- *$z_1, \dots, z_d$ are uniformly distributed and independent.*
- *For all $1 \le j \le d$, z_j is absolutely continuous. The common density of z_j and z_ℓ is bounded and Riemann integrable for all $j \ne \ell$.*
- *For all $1 \le j \le d$, z_j is trivial, i.e. $z_j \equiv 1$, and all zeros of f_j are roots of unity. Furthermore, $x_1, \dots, x_d$ are pairwise of finite type (see Definition 3.2.18).*

- *For all* $1 \leq j \leq d$, *there exists a* $\rho_j > 0$ *with* $(z_j)^{\rho_j} \equiv 1$, *all zeros of* f_j *are roots of unity and* $x_1, \ldots, x_d$ *are pairwise of finite type.*

We then have, as $n \to \infty$,

$$\frac{1}{\sqrt{\log n}} \begin{pmatrix} w^{1,n}(f_1)(x_1) \\ \vdots \\ w^{1,n}(f_d)(x_d) \end{pmatrix} - \theta \sqrt{\log n} \begin{pmatrix} m(f_1) \\ \vdots \\ m(f_d) \end{pmatrix} \xrightarrow{d} N = \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix},$$

where N *is a* $d-$*variate complex normal distributed random variable with, for* $j \neq \ell$,

$$\operatorname{Cov}\big(\operatorname{Re}(N_j), \operatorname{Re}(N_\ell)\big) = \theta \int_{[0,1]^2} \log\big|f_j(e^{2\pi i u})\big| \log\big|f_\ell(e^{2\pi i v})\big| \, dudv, \tag{3.4.54}$$

$$\operatorname{Cov}\big(\operatorname{Re}(N_j), \operatorname{Im}(N_\ell)\big) = \theta \int_{[0,1]^2} \log\big|f_j(e^{2\pi i u})\big| \arg\big(f_\ell(e^{2\pi i v})\big) \, dudv, \tag{3.4.55}$$

$$\operatorname{Cov}\big(\operatorname{Im}(N_j), \operatorname{Im}(N_\ell)\big) = \theta \int_{[0,1]^2} \arg\big(f_j(e^{2\pi i u})\big) \arg\big(f_\ell(e^{2\pi i v})\big) \, dudv. \tag{3.4.56}$$

The variance is given by

$$\operatorname{Var} \operatorname{Re}(N_j) = \theta \int_{[0,1]} \log^2\big|f_j(e^{2\pi i u})\big| \, du \tag{3.4.57}$$

and

$$\operatorname{Var} \operatorname{Im}(N_j) = \theta \int_{[0,1]} \arg^2\big(f_j(e^{2\pi i v})\big) \, dv. \tag{3.4.58}$$

Theorem 3.4.6. *Let* S_n *be endowed with the Ewens distribution with parameter* θ, $f_1, \ldots, f_d$ *be non zero real analytic functions,* $\overline{z} = (z_1, \ldots, z_d)$ *a* $\mathbb{T}^d$*-valued random variable and* $x_1 = e^{2\pi i \varphi_1}, \ldots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ *be such that* $1, \varphi_1, \ldots, \varphi_d$ *are linearly independent over* $\mathbb{Z}$.
Suppose that one of the following conditions is satisfied:

- $z_1, \ldots, z_d$ *are uniformly distributed and independent.*
- *For all* $1 \leq j \leq d$, z_j *is absolutely continuous. For each* $j \neq \ell$, *the joint density* $g_{j,\ell}$ *of* z_j *and* z_ℓ *satisfies*

$$g_{j,\ell}(\phi_j, \phi_\ell) = \sum_{a,b \in \mathbb{Z}} c_{a,b} e^{2\pi i (a\phi_j + b\phi_\ell)} \quad \text{and} \quad \sum_{a,b \in \mathbb{Z}} |c_{a,b}| < \infty. \tag{3.4.59}$$

- *For all* $1 \leq j \leq d$, z_j *is trivial, i.e.* $z_j \equiv 1$, *and all zeros of* f_j *are roots of unity. Furthermore,* $x_1, \ldots, x_d$ *are pairwise of finite type (see Definition 3.2.18),*

- *For all $1 \leq j \leq d$, z_j is discrete, there exists a $\rho_j > 0$ with $(z_j)^{\rho_j} \equiv 1$, all zeros of f_j are roots of unity. Furthermore, assume that $x_1, \ldots, x_d$ are pairwise of finite type (see Definition 3.2.18) and that for $j \neq \ell$*

$$\mathbb{P}\left[z_j = e^{2\pi i k_1/\rho_j}, z_\ell = e^{2\pi i k_2/\rho_\ell}\right] = \frac{1}{\rho_j \rho_\ell} \sum_{a=0}^{\rho_j - 1} \sum_{b=0}^{\rho_\ell - 1} c_{a,b} e^{2\pi i (a k_1 + b k_2)} \tag{3.4.60}$$

$$\text{with} \sum_{a=0}^{\rho_j-1} |c_{a,b}| < 1, \text{ for } b \neq 0 \quad \text{and} \sum_{b=0}^{\rho_\ell-1} |c_{a,b}| < 1, \text{ for } a \neq 0. \tag{3.4.61}$$

We then have, as $n \to \infty$,

$$\frac{1}{\sqrt{\log n}} \begin{pmatrix} w^{2,n}(f_1)(x_1) \\ \vdots \\ w^{2,n}(f_d)(x_d) \end{pmatrix} - \theta \sqrt{\log n} \begin{pmatrix} m(f_1) \\ \vdots \\ m(f_d) \end{pmatrix} \xrightarrow{d} N = \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $m(f_j)$ and N as in Theorem 3.4.5.

As before, we get as simple corollary, which covers Proposition 3.1.2:

Corollary 3.4.6.1. *Let S_n be endowed with the Ewens distribution with parameter θ, $\overline{z} = (z_1, \ldots, z_d)$ be a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i \varphi_1}, \ldots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \ldots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*
Suppose that one of the conditions in Theorem 3.4.6 is satisfied: We then have, as $n \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi^2}{12} \theta \log n}} \begin{pmatrix} \log\big(Z_{n,z_1}(x_1)\big) \\ \vdots \\ \log\big(Z_{n,z_d}(x_d)\big) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\mathrm{Re}(N_1), \ldots, \mathrm{Re}(N_d), \mathrm{Im}(N_1), \ldots, \mathrm{Im}(N_d)$ independent standard normal distributed random variables.

Proof of Theorem 3.4.5 and 3.4.6. We consider $w^{1,n}(f)$ and $w^{2,n}(f)$ as $\mathbb{R}^2$-valued random variables and argue with Theorem 3.3.4. The assumptions of Theorem 3.3.1 and Theorem 3.3.4 are almost the same. The only difference lies in condition (3.3.4) in Theorem 3.3.4:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[X_{m,1,j} X_{m,1,\ell}\right] = \sigma_{j,\ell}. \tag{3.4.62}$$

The computations for uniformly distributed and for absolute continuous $z_1, \ldots, z_d$ are for both, $w^{1,n}$ and $w^{2,n}$ the same as in Section 3.4.1, namely in Section 3.4.1.1 and Section 3.4.1.2. Thus, we have a closer look at the third and the forth condition in Theorem 3.4.5 and 3.4.6, the trivial and the discrete case. The behavior in one point, where $z \equiv 1$ has been treated by [40]. For the behavior at different points, we have to extend Lemma 3.4.4:

Lemma 3.4.7. *Let $f_1, f_2 : \mathbb{T} \to \mathbb{C}$ be real analytic with only roots of unity as zeros and let $x_1 = e^{2\pi i \varphi_1}$ and $x_2 = e^{2\pi i \varphi_2}$ be such that $(x_1, x_2) \in \mathbb{T}^2$ be of finite type (see Definition 3.2.16). We then have $\log\big|f_j(x_j^n)\big| = O\big(\log n\big)$ for $j \in \{1, 2\}$. Moreover, as $n \to \infty$*

$$\frac{1}{n}\sum_{m=1}^{n} \log|f_1(x_1^m)| \log|f_2(x_2^m)| \longrightarrow \int_{[0,1]^2} \log\big|f_1(e^{2\pi iu})\big| \log\big|f_2(e^{2\pi iv})\big| \, dudv, \tag{3.4.63}$$

$$\frac{1}{n}\sum_{m=1}^{n} \arg(f_1(x_1^m)) \log|f_2(x_2^m)| \longrightarrow \int_{[0,1]^2} \arg\big(f_1(e^{2\pi iu})\big) \log\big|f_2(e^{2\pi iv})\big| \, dudv \tag{3.4.64}$$

and

$$\frac{1}{n}\sum_{m=1}^{n} \arg(f_1(x_1^m)) \arg(f_2(x_2^m)) \longrightarrow \int_{[0,1]^2} \arg\big(f_1(e^{2\pi iu})\big) \arg\big(f_2(e^{2\pi iv})\big) \, dudv. \tag{3.4.65}$$

By using Lemma 3.4.7, the proof for $z_1, \ldots, z_d$ being discrete is the same as in Section 3.4.1.3. Thus, in order to conclude the proofs for Theorem 3.4.5 and 3.4.6, we will proceed by giving the proof of Lemma 3.4.7:

Proof. We start by considering (3.4.63). Since x_1 and x_2 are not roots of unity, we expect for $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n} \log|f_1(x_1^m)| \log|f_2(x_2^m)| \longrightarrow \int_{[0,1]^2} \log\big|f_1(e^{2\pi iu})\big| \log\big|f_2(e^{2\pi iv})\big| \, dudv. \tag{3.4.66}$$

Unfortunately this is not automatically true since $\log(f_j)$ is not Riemann integrable and we thus cannot apply Theorem 3.2.6. We show here that (3.4.66) is true by using the assumption that (x_1, x_2) is of finite type.
We use the notations:

$$\begin{aligned} h_1(\phi) &:= \log|f_1(e^{2\pi i\phi})|, \ \varphi_1^{(m)} := \{m\varphi_1\}, \ \boldsymbol{\varphi}_1 := (\varphi_1^{(m)})_{m=1}^{\infty}, \\ h_2(\phi) &:= \log|f_2(e^{2\pi i\phi})|, \ \varphi_2^{(m)} := \{m\varphi_2\}, \ \boldsymbol{\varphi}_2 := (\varphi_2^{(m)})_{m=1}^{\infty}, \\ \overline{\varphi} &:= (\varphi_1, \varphi_2), \ \varphi^{(m)} := (\varphi_1^{(m)}, \varphi_2^{(m)}) \ \boldsymbol{\varphi} := \big(\varphi^{(m)}\big)_{m=1}^{\infty}. \end{aligned} \tag{3.4.67}$$

We thus can reformulate the LHS of (3.4.63) as

$$\frac{1}{n}\sum_{m=1}^{n} h_1(\varphi_1^{(m)}) h_2(\varphi_2^{(m)}). \tag{3.4.68}$$

If f_1 and f_2 are zero free, then h_1 and h_2 are Riemann integrable. Furthermore, $1, \varphi_1, \varphi_2$ are by assumption linearly independent over $\mathbb{Z}$, and thus $\boldsymbol{\varphi}$ is a uniformly distributed sequence by Lemma 3.2.14. Equation (3.4.66) now follows immediately with Theorem 3.2.6.

If f_1 and f_2 are not zero free, we have to be more careful. We use in this case Theorem 3.2.13 for $d = 2$. We assume for simplicity that 0 and 1 are to the only singularities of h_1 and h_2. The more general case with roots of unity as zeros is completely similar.
We first have to choose a suitable $\delta = \delta(n)$ such that $\varphi^{(m)} \in [\delta, 1-\delta]^2$ for $1 \leq m \leq n$. Since by assumption $\overline{\varphi}$ is of finite type, there exists $K > 0, \gamma > 1$ such that

$$\|\overline{q} \cdot \overline{\varphi}\| \geq \frac{K}{(\|\overline{q}\|_\infty)^\gamma} \quad \text{for all } \overline{q} \in \mathbb{Z}^2 \setminus \{0\} \tag{3.4.69}$$

with $\|a\| := \inf_{m \in \mathbb{Z}} |a - m|$. We thus can chose $\delta = \frac{K}{n^\gamma}$.
Next, we have to estimate the discrepancies of the sequences $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2$ and $\boldsymbol{\varphi}$. Since $\overline{\varphi}, \varphi_1, \varphi_2$ are of finite type, we can use Theorem 3.2.17 and get

$$D_n^*(\boldsymbol{\varphi}_1) = O(n^{-\alpha_1}), \ D_n^*(\boldsymbol{\varphi}_2) = O(n^{-\alpha_2}) \ \text{ and } \ D_n^*(\boldsymbol{\varphi}) = O(n^{-\alpha}) \tag{3.4.70}$$

for some $\alpha_1, \alpha_2, \alpha > 0$.
We can show now with Theorem 3.2.13 that the error made by the approximation in (3.4.66) goes to 0 by showing that all summands on the RHS of (3.2.16) go to 0. This computation is straight forward and very similar for each summand. We restrict ourselves to illustrate the computations only on the summands corresponding to the face F of $[\delta, 1-\delta]^2$ with $\phi_1 = 1 - \delta$. We get with $h(\phi_1, \phi_2) := h_1(\phi_1)h_2(\phi_2)$,

$$\delta |h_1(1-\delta)| \int_\delta^{1-\delta} |h_2(u)| \, du + D_n^*(\boldsymbol{\varphi}_2) |h_1(1-\delta)| V(h_2|[\delta, 1-\delta]), \tag{3.4.71}$$

where $V(h_2|[\delta, 1-\delta])$ is the variation of $h_2|[\delta, 1-\delta]$. It is easy to see that, for $\phi \to 0$ and some $K_1 > 0$, $h_1(\phi) \sim K_1 \log(\phi) \sim h_1(1-\phi)$. Thus, the first summand in (3.4.71) goes to 0 for $n \to \infty$. On the other hand we have

$$D_n^*(\boldsymbol{\varphi}_2) |h_1(1-\delta)| V(h_2|[\delta, 1-\delta]) \sim K_2 D_n^*(\boldsymbol{\varphi}_2) \log^2 \delta \leq K_3 n^{-\alpha_2} \log^2 n. \tag{3.4.72}$$

for constants $K_2, K_3 > 0$. This shows that also the second term in (3.4.71) goes to 0. So, we proved (3.4.63). Equations (3.4.64) and (3.4.65) are straightforward, with the given computations above and we conclude Lemma 3.4.7. □

This completes the proofs of Theorem 3.4.5 and 3.4.6. □

CHAPTER 4

Expansion on fluctuations of linear statistics

As a natural continuation of the work by Ben Arous and Dang [10] and Dang and Zeindler [19], we present here results on linear statistics for matrix models closely related to random permutations, sampled under the Ewens measure. The proofs are direct applications of the tools given in [10]. The results are closely related to results obtained in a recent paper by Hughes, Najnudel, Nikeghbali and Zeindler [43]

4.1 Introduction

The study of global fluctuations of random matrices is a major field in Random Matrix Theory (RMT) (see for example [6], [7], [8], [10], [14], [16], [17], [19], [21], [23], [24], [25], [27], [45], [46], [47], [49], [50], [51], [54], [59], [60], [63], [66], [68] or [77]). However, the study of random permutation matrices has been playing a rather subordinated role, compared to the study of Wigner matrices or classical invariant ensembles. As a finite subgroup of unitary matrices, the spectrum of random permutation matrices follow in general the behavior of ensembles studied in RMT (see for instance [19], [22], [40], [76], [78], [82]). This is not obvious and one could expect a very different behavior for the non-invariant ensemble of random permutation matrices due to the distinctive eigenvalue structure. Indeed, a recent result by Ben Arous and Dang [10] shows that linear statistics of random permutation matrices admit a very different behavior as linear statistics of other matrix ensembles, if the function is smooth enough. We will see that this "non-universal" behavior naturally applies for matrices closely related to random permutations.
In [57], Nikeghbali and Najnudel introduced a matrix model based on random permutations. In particular, they considered the following matrix ensemble:
Let $\sigma \in \mathcal{S}_N$ be a permutation chosen at random from the Ewens distribution with paramter $\theta > 0$. Consider matrices

$$M_{\sigma,z} = (z_i \delta_{i\sigma(j)})_{1\le i,j\le N}, \tag{4.1.1}$$

where the random variables z_i, $1 \le i \le N$, are i.i.d. random variables in the complex plane $\mathbb{C}^*$.

This ensemble is a subgroup of the linear subgroup $GL(N,\mathbb{C}^*)$. The classical random permutation matrix ensemble, i.e. permutation matrices, where the permutations are chosen uniformly from the symmetric group, correspond to $\theta = 1$ and by choosing z_i, $1 \le i \le N$, with Dirac mass at 1. Matrices $M_{\sigma,z}$ can also be seen as the wreath product of $\mathbb{C}^*$ and $\mathcal{S}_N$. If the random variables z_i, $1 \le i \le N$, have values only on the unit circle, then matrices $M_{\sigma,z}$ form a subgroup of the group of unitary matrices. A main motivation of studying matrices $M_{\sigma,z}$ is that the spectrum gives "more" parallels to classical matrix ensembles such as Dyson's Circular Ensembles, but still keep the eigenvalue structure of permutation matrices. Eigenvalues of permutation matrices can be simply obtained by the cycle structure of the permutation σ, where asymptotics are well-known (see for instance [2], [3], [4] or [5]).

Recently, Dang and Zeindler showed central limit theorems for the characteristic polynomial of matrices $M_{\sigma,z}$, where the values $z_i \overset{d}{=} z$ are restricted to lie on the unit circle [19]. We will show for this matrix model that linear statistics behave as given in [10], by applying the tools of [10]. Similar results has been obtained by [43].

We start by giving some notations. For a periodic function f on the interval $[0,1]$ and $f(x) = \tilde{f}(e^{2i\pi x})$, define the linear statistic

$$I_{\sigma,N,z}(f) = \sum_{i=1}^{N} \tilde{f}(\lambda_i) = \sum_{i=1}^{N} f(\varphi_i), \tag{4.1.2}$$

where $\lambda_1 = e^{2i\pi\varphi_1}, \ldots, \lambda_N = e^{2i\pi\varphi_N}$ are the eigenvalues of $M_{\sigma,z}$. It is easy to see that for any cycle of σ of length j, the corresponding j eigenvalues are of the form

$$T_j \cdot \omega, \tag{4.1.3}$$

where $\omega^j = 1$ and T_j is defined by

$$T_j = e^{2i\pi\psi_j} \overset{d}{=} \left(\prod_{k=1}^{j} z_k\right)^{1/j}. \tag{4.1.4}$$

For every $1 \le j \le N$, define α_j to be the number of cycles of length j of σ. Note that, for each $1 \le j \le N$, we have α_j different i.i.d. random variables

$$T_j^{(1)}, \ldots, T_j^{(\alpha_j)}. \tag{4.1.5}$$

This gives the $j \cdot \alpha_j$ i.i.d. eigenangles for each $1 \le j \le N$:

$$\frac{k}{j} + \psi_j^{(1)}, \frac{k}{j} + \psi_j^{(2)}, \ldots, \frac{k}{j} + \psi_j^{(\alpha_j)}, \quad k = 0, \ldots, j-1 \tag{4.1.6}$$

The linear statistic $I_{\sigma,N,z}(f)$ is then equal in law to

$$\sum_{j=1}^{N} \sum_{\ell=1}^{\alpha_j} \left[\sum_{k=0}^{j-1} f(\frac{k}{j} + \psi_j^{(\ell)})\right] = \sum_{j=1}^{N} \sum_{\ell=1}^{\alpha_j} \sum_{k=0}^{j-1} \tau_{j,\ell} f(\frac{k}{j}), \tag{4.1.7}$$

where τ denotes the translation of f by a random factor ψ:

$$\tau_{j,\ell} f(x) = f(x + \psi_j^{(\ell)}). \tag{4.1.8}$$

We study here the asymptotic behavior of the linear statistic $I_{\sigma,N,z}(f)$, where σ is chosen with respect to the Ewens distribution for any parameter $\theta > 0$, independent of the random variable z with given law $\mathcal{L}$ and a wide class of functions f. The asymptotic behavior depends strongly on the smoothness of f and the limit law depends on the law $\mathcal{L}$. In cases where f is smooth enough so that the variance of $I_{\sigma,N,z}$ is finite, the centered linear statistic converges in law to an infinitely divisible distribution. We will give the Lévy measure explicitly below. For f being less smooth, we obtain a central limit theorem for $I_{\sigma,N,z}$. The proofs follow the same structure and using the same tools as the proofs for the results in [10].

We start by quantifying the linear statistic $I_{\sigma,N,z}$ in terms of the sequence

$$R_{j,\ell}(f) = \frac{1}{j}\sum_{k=0}^{j-1} \tau_{j,\ell} f\left(\frac{k}{j}\right) - \int_0^1 \tau_{j,\ell} f(x)dx = \frac{1}{j}\sum_{k=0}^{j-1} \tau_{j,\ell} f\left(\frac{k}{j}\right) - \int_0^1 f(x)dx, \tag{4.1.9}$$

where f is periodic on $[0,1]$. Clearly,

$$R_{j,\ell}(f) = R_j(\tau_{j,\ell} f), \tag{4.1.10}$$

where $R_j(f)$ is the error in the composite trapezoidal approximation to the integral of f (see [10], [20]).

Expanding (4.1.7) in terms of $R_{j,\ell}(f)$, we see that $I_{\sigma,N,z}$ is controlled by the asymptotic behavior of $R_{j,\ell}(f)$. In particular, $I_{\sigma,N,z}$ is equal in law to

$$N\int_0^1 f(x)dx + \sum_{j=1}^{N}\sum_{\ell=1}^{\alpha_j} jR_{j,\ell}(f). \tag{4.1.11}$$

Note that $R_{j,\ell}(f)$ is for all j,ℓ independent and for fixed j, the $R_{j,\ell}(f)$'s are i.i.d. random variables, so that we sometimes write

$$\hat{R}_j(f) := R_{j,\ell}(f) \tag{4.1.12}$$

whenever the role of ℓ can be omitted. For the rest of the paper, we set

$$I_{\sigma,N,z}(f) = N\int_0^1 f(x)dx + \sum_{j=1}^{N}\sum_{\ell=1}^{\alpha_j} jR_{j,\ell}(f). \tag{4.1.13}$$

By (4.1.13), it becomes more clear why the smoothness of f plays an important role for the limit behavior of $I_{\sigma,N,z}$. If f is smooth enough, then clearly the random translation of f is also smooth enough and the sequence $R_{j,\ell}(f)$ converges fast enough to zero, so that $I_{\sigma,N,z}$ is well approximated by the first terms of the sum in

(4.1.13). If f is less smooth, then also the random translation of f is less smooth and the sum in (4.1.13) blows up (slowly) and a normalization is needed in order to get a limit law.

We present first a result for fluctuations of linear statistics for smooth functions, which generalizes Theorem 2.3 of [10].

Theorem 4.1.1. *For $\theta > 0$ and f being of bounded variation, assume that*

$$\sum_{j=1}^{\infty} j \sup_{j,z} \hat{R}_j^2(f) \in (0, \infty). \tag{4.1.14}$$

1. *Then the distribution of*

$$I_{\sigma,N,z}(f) - \mathbb{E}\left[I_{\sigma,N,z}(f)\right] \tag{4.1.15}$$

converges to a non-Gaussian infinitely divisible distribution μ, defined by

$$\log \hat{\mu} = \int (e^{itx} - 1 - itx) dM_f(x), \tag{4.1.16}$$

where ν_j is the distribution of $j\hat{R}_j(f)$ and the Lévy measure M_f is given by

$$M_f = \sum_{j=1}^{\infty} \frac{\theta}{j} \nu_j. \tag{4.1.17}$$

2. *The asymptotic behavior of the expectation of $I_{\sigma,N,z}$ is given by*

$$\mathbb{E}\left[I_{\sigma,N,z}(f)\right] = N \int_0^1 f(x)dx + \theta \sum_{j=1}^{N} \mathbb{E}\left[\hat{R}_j(f)\right] + o(1). \tag{4.1.18}$$

3. *The asymptotic behavior of the variance of $I_{\sigma,N,z}$ is given by*

$$\operatorname{Var} I_{\sigma,N,z}(f) = \theta \sum_{j=1}^{N} j\mathbb{E}\left[\hat{R}_j^2(f)\right] + o(1). \tag{4.1.19}$$

Remark 4.1.1.1. *We will prove a more general version of Theorem 4.1.1 in Section 4.3, requiring conditions on the Cesaro means. We will see that f being of bounded variation is not a necessary assumption. In particular, for $\theta \geq 1$, the condition given in* (4.1.14) *is necessary and sufficient.*

Remark 4.1.1.2. *The proof Theorem 4.1.1 is a direct application of the proof of Theorem 2.3 in [10]. Since arguments on f being of bounded variation are insensitive to translations τf (see below Lemma 4.3.4, Section 4.3), Theorem 4.1.1 applies as a direct consequence for functions treated in [10]. We state some examples below.*

Example 4.1.2. Assume that $\hat{R}_j(f)$ is not zero for at least one j or z, in order to avoid the sum in (4.1.14) to be zero. This is the case whenever the even part of f is not a constant (see [10]).Then the conclusions of Theorem 4.1.1 hold for $f \in \mathcal{C}^1$, so that the modulus of continuity $\omega(f', \delta)$ of the derivative of f satisfies

$$\sum_{j=1}^{\infty} \frac{1}{j} \omega(f', 1/j)^2 < \infty. \tag{4.1.20}$$

Of course, (4.1.20) is satisfied if $f \in \mathcal{C}^{1+\alpha}$, for $0 < \alpha < 1$, i.e. if f' is α-Hölder continuous.

Example 4.1.3. If f has a derivative in L^p, let $\omega^{(p)}(f', \delta)$ be the modulus of continuity in L^p of its derivative f', i.e.

$$\omega^{(p)}(f', \delta) = \sup_{0 \le h \le \delta} \left\{ \int_0^1 |f'(x+h) - f'(x)|^p \right\}^{1/p}. \tag{4.1.21}$$

Assume that

$$\omega^{(p)}(f', \delta) \le \delta^\alpha \quad \text{with } \alpha > \frac{1}{p}, \tag{4.1.22}$$

also assume that the even part of f is not a constant in order to avoid the trivial case mentioned above, then the conclusions of Theorem 4.1.1 hold.

Example 4.1.4. If f is in the Sobolev space H^s, for $s > 1$, and if one assumes that the even part of f is not a constant, then the conclusions of Theorem 2.1.1 hold.

Remark 4.1.4.1. *In a recent work by Hughes, Najnudel, Nikeghbali and Zeindler [43], the authors deduce a limiting infinitely divisible distribution of linear statistics for a more general class of probability measures on the symmetric group, where the cycles of different lengths are weighted. However, their proof differ substantially from the proof of Theorem 4.1.1, given in Section 4.3, since the Feller Coupling does not apply in their case. Instead, the proof in [43] requires combinatorial arguments and singular analysis of generating functions. The conditions so that the infinitely divisible limit distribution holds can be translated to conditions on the Fourier coefficients of f. They also hold for f being a Laurent polynomial or lying in a certain Sobolev space (see Theorem 4.4, Section 5 in [43]).*

We state next a central limit theorem result for fluctuations of linear statistics for less smooth functions, where the variance is infinite.

Theorem 4.1.5. *Let f be of bounded variation so that for any $\theta > 0$*

$$\sum_{j=1}^{\infty} j \inf_{j,z} \hat{R}_j^2(f) = \infty. \tag{4.1.23}$$

1. *Then,*

$$\frac{I_{\sigma,N,z}(f) - \mathbb{E}\left[I_{\sigma,N,z}(f)\right]}{\sqrt{\operatorname{Var} I_{\sigma,N,z}(f)}} \tag{4.1.24}$$

converges in distribution to $\mathcal{N}(0,1)$.

2. *The asymptotic behavior of the expectation is given by*

$$\mathbb{E}\left[I_{\sigma,N,z}(f)\right] = N\int_0^1 f(x)dx + \sum_{j=1}^{N} \theta\mathbb{E}\left[\hat{R}_j(f)\right] + O(1). \tag{4.1.25}$$

3. *The asymptotic behavior of the variance is given by*

$$\operatorname{Var} I_{\sigma,N,z}(f) \sim \sum_{j=1}^{N} \theta j \mathbb{E}\left[\hat{R}_j^2(f)\right]. \tag{4.1.26}$$

Remark 4.1.5.1. *In fact, f being of bounded variation is not a necessary condition. We will prove the conclusions of Theorem 4.1.5 with weaker conditions in Section 4.4. Indeed, we will replace the condition on f being of bounded variation by the* sup*-Norm of $j\hat{R}_j(f)$ being sufficiently small. Theorem 6.2 in [43] gives a similar result, applying the proof of [10] for infinite variance. Their condition slightly differs and is given in terms of a p-Norm instead of the* sup*-Norm.*

In Section 4.2, we recall some basic estimates which we will use in the proofs of Theorem 4.1.1 and Theorem 4.1.5. We will give the proof of Theorem 4.1.1 and a slightly modified version for $\theta > 0$ in Section 4.3. We prove Theorem 4.1.5 in Section 4.4.

4.2 Estimates on the Feller Coupling

We recall briefly the definition and some properties of the Feller coupling. We will make use of this tool to study the asymptotic behavior of α_j, the number of cycles of length j of a permutation σ (see for example [2], p. 523).
Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $(\xi_i)_{i\geq 1}$ of independent Bernoulli random variables defined on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}[\xi_i = 1] = \frac{\theta}{\theta + i - 1} \quad \text{and} \quad \mathbb{P}[\xi_i = 0] = \frac{i-1}{\theta + i - 1}.$$

For $1 \leq j \leq N$, denote the number of spacings of length j in the sequence $1\xi_2 \cdots \xi_N 1$ by $C_j(N)$, i.e.

$$C_j(N) = \sum_{i=1}^{N-j} \xi_i(1-\xi_{i+1})\dots(1-\xi_{i+j-1})\xi_{i+j} + \xi_{N-j+1}(1-\xi_{N-j+2})\dots(1-\xi_N). \tag{4.2.1}$$

Define $(Y_{jm})_{j\geq 1}$ by

$$Y_{jm} = \sum_{i=m+1}^{\infty} \xi_i(1-\xi_{i+1})\dots(1-\xi_{i+j-1})\xi_{i+j} \tag{4.2.2}$$

and set for $j \geq 1$,

$$Y_j := Y_{j0}. \tag{4.2.3}$$

Theorem 4.2.1. *Under the Ewens distribution with parameter θ,*

(i) $(C_j(N))_{1\le j\le N}$ *has the same distribution as* $(\alpha_j)_{1\le j\le N}$*, i.e. for any* $a=(a_1,\dots,a_N)\in\mathbb{N}^N$,

$$\mathbb{P}[(C_1(N),\dots,C_N(N)=a]=\nu_{N,\theta}[(\alpha_1(\sigma),\dots,\alpha_N(\sigma)=a]. \tag{4.2.4}$$

(ii) $(Y_j)_{1\le j\le N}$ *are independent Poisson random variables with mean* θ/j.

Recall that

$$I_{\sigma,N,z}(f)=N\int_0^1 f(x)dx+\sum_{j=1}^N\sum_{\ell=1}^{\alpha_j} jR_{j,\ell}(f). \tag{4.2.5}$$

In order to study $I_{\sigma,N,z}(f)$, define the random variables

$$G_{\sigma,N,z}=\sum_{j=1}^N\sum_{\ell=1}^{C_j(N)} jR_{j,\ell}(f) \tag{4.2.6}$$

and

$$H_{N,z}=\sum_{j=1}^N\sum_{\ell=1}^{Y_j} jR_{j,\ell}(f). \tag{4.2.7}$$

Obviously,

$$\mathbb{E}\left[|G_{\sigma,N,z}-H_{N,z}|\right]\le\sum_{j=1}^N\mathbb{E}\left[|C_j(N)-Y_j|\right]\mathbb{E}\left[|jR_{j,1}(f)|\right] \tag{4.2.8}$$

and

$$\begin{aligned}&\mathbb{E}\left[(G_{\sigma,N,z}-H_{N,z})^2\right]\\ &\quad\le\sum_{j=1}^N\sum_{k=1}^N\mathbb{E}\left[|C_j(N)-Y_j|\right]\mathbb{E}\left[|C_k(N)-Y_k|\right]\mathbb{E}\left[|jR_{j,1}(f)|\right]\mathbb{E}\left[|kR_{k,1}(f)|\right]\\ &.\end{aligned} \tag{4.2.9}$$

Define for $\gamma=\theta-1$

$$\Psi_N(j)=\binom{N-j+\gamma}{N-j}\binom{N+\gamma}{N}^{-1}=\prod_{k=0}^{j-1}\frac{N-k}{\theta+N-k-1} \tag{4.2.10}$$

Lemma 4.4., p. 15 in [10], shows the following:

Lemma 4.2.2 (Ben Arous-Dang)**.** *For any $\theta>0$ there exists a constant $C(\theta)$ such that for every N*

$$\mathbb{E}\left[|C_j(N)-Y_j|\right]\le\frac{C(\theta)}{N}+\frac{\theta}{N}\Psi_N(j). \tag{4.2.11}$$

Identifying $(|u_j|)_{j\geq 1}$ with $(\mathbb{E}\left[|j\hat{R}_j(f)|\right])_{j\geq 1}$, we have the following L^1 bound by Lemma 4.2.2 (see also Lemma 4.3, [10])

Lemma 4.2.3 (Ben Arous-Dang). *For any $\theta > 0$ there exists a constant $C(\theta)$ such that for every N*

$$\mathbb{E}\left[|G_{\sigma,N,z} - H_{N,z}|\right] \leq \frac{C(\theta)}{N}\sum_{j=1}^{N}|u_j| + \frac{\theta}{N}\sum_{j=1}^{N}|u_j|\Psi_N(j). \tag{4.2.12}$$

Define $u_j^2 = \mathbb{E}\left[j^2\hat{R}_j^2(f)\right]$. The L^2 bound is given by the following Lemma (see [10], p.16):

Lemma 4.2.4 (Ben Arous-Dang). *For every $\theta > 0$, there exists a constant $C(\theta)$ such that, for every integer N,*

$$\begin{aligned}&\mathbb{E}\left(G_{\sigma,N,z} - H_{N,z})^2\right)\\ &\leq C(\theta)\left[(\frac{1}{N}\sum_{j=1}^{N}|u_j|)^2 + \frac{1}{N}\sum_{j=1}^{N}|u_j|^2 + \frac{1}{N^2}\sum_{j=1}^{N}|u_j|\sum_{k=1}^{N}|u_k|\Psi_N(k) + \frac{1}{N}\sum_{j=1}^{N}|u_j|^2\Psi_N(j)\right]\end{aligned} \tag{4.2.13}$$

Lemma 4.2.3 and Lemma 4.2.4 imply the following result:

Lemma 4.2.5. *For every $\theta > 0$, there exist constants $C_1(\theta)$ and $C_2(\theta)$ such that, for every integer N,*

(i)

$$\mathbb{E}\left[|G_{\sigma,N,z} - H_{N,z}|\right] \leq C_1(\theta)\sup_{1\leq j\leq N}|jR_{j,1}(f)| \tag{4.2.14}$$

(ii)

$$\mathbb{E}\left[(G_{\sigma,N,z} - H_{N,z})^2\right] \leq C_2(\theta)\sup_{1\leq j\leq N}(jR_{j,1}(f))^2 \tag{4.2.15}$$

Proof. This is an immediate consequence of the fact (see [10], p. 15) that

$$\frac{\theta}{N}\sum_{j=1}^{N}\Psi_N(j) = 1. \tag{4.2.16}$$

□

In order to extend the conditions of Theorem 4.1.1 to any $\theta > 0$, we introduce the terms of Cesaro means. In fact, Lemma 4.2.3 and Lemma 4.2.4 can be interpreted in terms of Cesaro means of sequences $(u_j)_{j\geq 1}$. We recall briefly the following result, which links the Cesaro mean to the Feller Coupling (see [10], p. 20).

Lemma 4.2.6. *The Cesaro mean σ_N^θ of order θ of a sequence $s = (s_j)_{j\geq 0}$, with $s_0 = 0$, is given by*

$$\sigma_N^\theta(s) = \frac{\theta}{N+\theta}\sum_{j=1}^{N} s_j \Psi_N(j). \tag{4.2.17}$$

For $u = (u_j)_{j\geq 1}$, restating Lemma 4.2.3 and Lemma 4.2.4 gives the following:

Lemma 4.2.7. *For an $\theta > 0$, there exist constants $C_1(\theta)$ and $C_2(\theta)$ such that*

(i)

$$\mathbb{E}\left[|G_{\sigma,N,z} - H_{N,z}|\right] \leq C_1(\theta)(\sigma_N^1(|u|) + \sigma_N^\theta(|u|)) \tag{4.2.18}$$

(ii) and

$$\mathbb{E}\left[(G_{\sigma,N,z} - H_{N,z})^2\right] \leq C_2(\theta)[\sigma_N^1(|u|)^2 + \sigma_N^1(u^2) + \sigma_N^1(|u|)\sigma_N^\theta(|u|) + \sigma_N^\theta(u^2)]. \tag{4.2.19}$$

We are now ready to prove the results.

4.3 A Non-Gaussian Limit

In Section 4.2, we introduced the tools needed to prove the following generalization of Theorem 4.1.1.

Theorem 4.3.1. *Assume that the sequence $(j\mathbb{E}\left[|\hat{R}_j(f)|\right])_{j\geq 1}$ converges in Cesaro (C,θ) to zero if $\theta < 1$ and assume that*

$$\sum_{j=1}^{\infty} j\mathbb{E}\left[\hat{R}_j^2(f)\right] \in (0,\infty). \tag{4.3.1}$$

Then,

1. *the distribution of*

$$I_{\sigma,N,z} - \mathbb{E}\left[I_{\sigma,N,z}\right] \tag{4.3.2}$$

converges to an infinitely divisible distribution μ, defined by

$$\log\hat{\mu} = \sum_{j=1}^{\infty}\frac{\theta}{j}\left(\hat{\nu}_j(t) - 1 - itm_j\right), \tag{4.3.3}$$

where ν_j is the distribution of $j\hat{R}_j(f)$ and m_j its expectation.

2. *The asymptotic behavior of the expectation of $I_{\sigma,N,z}$ is given by*

$$\mathbb{E}\left[I_{\sigma,N,z}\right] = N\int_0^1 f(x)dx + \theta\sum_{j=1}^{N}\mathbb{E}\left[\hat{R}_j(f)\right] + o(1). \tag{4.3.4}$$

3. *Assume additionally that* $(j^2\mathbb{E}\left[|\hat{R}_j^2(f)|\right])_{j\geq 1}$ *converges in Cesaro* $(C,\theta\wedge 1)$. *Then, the asymptotic behavior of the variance of* $I_{\sigma,N,z}$ *is given by*

$$\operatorname{Var} I_{\sigma,N,z} = \theta\sum_{j=1}^{N} j\mathbb{E}\left[\hat{R}_j^2(f)\right] + o(1). \tag{4.3.5}$$

Recall that

$$H_{N,z} = \sum_{j=1}^{N}\sum_{\ell=1}^{Y_j} jR_{j,\ell}(f), \tag{4.3.6}$$

where $(Y_j)_{j\geq 1}$ is a sequence of independent Poisson random variables with mean θ/j. We will show that the centered limit law of $H_{N,z}$ is given by μ and conclusions of Theorem 4.3.1 hold if $H_{N,z}$ and $I_{\sigma,N,z}$ behave asymptotically similar.

We start by giving the expectation and the variance of the random variable $H_{N,z}$.

Simple computation shows that:

$$\mathbb{E}\left[H_{N,z}\right] = \sum_{j=1}^{N}\mathbb{E}\left[Y_j\right]m_j = \sum_{j=1}^{N}\frac{\theta}{j}m_j, \tag{4.3.7}$$

where $m_j = m_j(f) = \mathbb{E}\left[j\hat{R}_j(f)\right]$. Define $V_j = \sum_{\ell=1}^{Y_j} jR_{j,\ell}(f)$, so that

$$\begin{aligned}
\operatorname{Var} H_{N,z} &= \sum_{j=1}^{N}\mathbb{E}\left[\operatorname{Var} V_j|Y_j\right] + \operatorname{Var}\mathbb{E}\left[V_j|Y_j\right] \\
&= \sum_{j=1}^{N}\mathbb{E}\left[Y_j\mathbb{E}\left[j^2\hat{R}_j^2(f)\right] - Y_j m_j^2\right] + \operatorname{Var} Y_j m_j \\
&= \sum_{j=1}^{N}\frac{\theta}{j}j^2\mathbb{E}\left[\hat{R}_j^2(f)\right] - \frac{\theta}{j}m_j^2 + m_j^2\operatorname{Var} Y_j \\
&= \sum_{j=1}^{N}\theta j\mathbb{E}\left[\hat{R}_j^2(f)\right] - \frac{\theta}{j}m_j^2 + m_j^2\frac{\theta}{j} \\
&= \sum_{j=1}^{N}\theta j\mathbb{E}\left[\hat{R}_j^2(f)\right].
\end{aligned}\tag{4.3.8}$$

Obviously, condition (4.3.1) is a condition on the variance of $H_{N,z}$.

Next, we prove the following lemma.

Lemma 4.3.2. *Let* ν_j *be the distribution of* $j\hat{R}_j(f)$ *and* m_j *its expectation. The distribution* μ_N *of* $H_{N,z} - \mathbb{E}\left[H_{N,z}\right]$ *satisfies*

$$\log\hat{\mu}_N = \sum_{j=1}^{N}\frac{\theta}{j}\left(\hat{\nu}_j(t) - 1 - itm_j\right), \tag{4.3.9}$$

Proof. Define for a fixed j

$$V_j := \sum_{\ell=1}^{Y_j} j R_{j,\ell}(f), \tag{4.3.10}$$

then

$$\begin{aligned} \mathbb{E}\left[e^{it(V_j - \mathbb{E}[V_j])}\right] &= e^{-it\mathbb{E}[V_j]}\mathbb{E}\left[e^{itV_j}\right] = \exp\left(\frac{it\theta m_j}{j}\right)\mathbb{E}\left[\hat{\nu}_j(t)^{Y_j}\right] && (4.3.11)\\ &= \exp\left(\frac{it\theta m_j}{j}\right)\exp\left(\frac{\theta}{j}(\hat{\nu}_j(t)-1)\right), && (4.3.12) \end{aligned}$$

which gives

$$\log \mathbb{E}\left[e^{it(V_j - \mathbb{E}[V_j])}\right] = \frac{\theta}{j}\left(\hat{\nu}_j(t) - itm_j - 1\right). \tag{4.3.13}$$

By the independence of all $R_{j,\ell}(f)$'s,

$$\log \mathbb{E}\left[e^{it(H_{N,z} - \mathbb{E}[H_{N,z}])}\right] = \sum_{j=1}^{N} \frac{\theta}{j}\left(\hat{\nu}_j(t) - itm_j - 1\right), \tag{4.3.14}$$

which proves Lemma 4.3.2 □

The following lemma is a result of convergence in distribution for the random variable $H_{N,z}$:

Lemma 4.3.3. *Under the assumption* (4.3.1)*, i.e.*

$$\sum_{j=1}^{\infty} j\mathbb{E}\left[\hat{R}_j^2(f)\right] \in (0, \infty), \tag{4.3.15}$$

the distribution μ_N of $H_{N,z} - \mathbb{E}[H_{N,z}]$ converges weakly to an infinitely divisible distribution μ defined by its Fourier transform

$$\log \hat{\mu}_N = \sum_{j=1}^{N} \frac{\theta}{j}\left(\hat{\nu}_j(t) - 1 - itm_j\right), \tag{4.3.16}$$

i.e.

$$\log \hat{\mu} = \int (e^{itx} - 1 - itx) dM_f(x), \tag{4.3.17}$$

with Lévy measure M_f given by

$$M_f = \sum_{j=1}^{\infty} \frac{\theta}{j}\nu_j. \tag{4.3.18}$$

Proof. Obviously, for $|t| \leq T$,

$$\left|\frac{\theta}{j}\left(\hat{\nu}_j(t) - itm_j - 1\right)\right| \leq \frac{\theta}{j}\frac{t^2}{2}\mathbb{E}\left[(j\hat{R}_j(f))^2\right] \leq \frac{\theta T^2}{2} j\mathbb{E}\left[\hat{R}_j^2(f)\right]. \tag{4.3.19}$$

By (4.3.1), $\log \widehat{\mu}_N(t)$ converges absolutely uniformly and its limit

$$\sum_{j=1}^{\infty} \frac{\theta}{j}(\hat{\nu}_j(t) - itm_j - 1) \tag{4.3.20}$$

is continuous. By Lévy's Theorem, the exponential of (4.3.20) is the Fourier transform of the probability measure μ and μ_N converges in distribution to μ as N goes to infinity.

Moreover, μ_N has a Lévy-Khintchine representation in terms of (a_N, M_N, σ_N^2), where $\sigma_N^2 = 0$,

$$M_N = \sum_{j=1}^{N} \frac{\theta}{j}\nu_j \tag{4.3.21}$$

and

$$a_N = \sum_{j=1}^{N} \frac{\theta}{j}\left(\mathbb{E}\left[\frac{j\hat{R}_j(f)}{1 + j^2\hat{R}_j(f)^2}\right] - \mathbb{E}\left[j\hat{R}_j(f)\right]\right) = \sum_{j=1}^{N} \frac{\theta}{j}\mathbb{E}\left[\frac{-j^3\hat{R}_j(f)^3}{1 + j^2\hat{R}_j(f)^2}\right]. \tag{4.3.22}$$

Clearly, the measure M_N is admissible by assumption (4.3.1), i.e.

$$\int x^2 dM_N(x) = \sum_{j=1}^{N} \theta j\mathbb{E}\left[\hat{R}_j^2(f)\right] < \infty, \tag{4.3.23}$$

which proves Lemma 4.3.3.

□

We continue by proving Theorem 4.3.1. Recall that the random variable $G_{\sigma,N,z}$ is defined by

$$\sum_{j=1}^{N} \sum_{\ell=1}^{C_j(N)} jR_{j,\ell}(f). \tag{4.3.24}$$

Proof of Theorem 4.3.1. We start by proving the first point of Theorem 4.3.1:
If $\theta \geq 1$, then

$$\sum_{j=1}^{\infty} j\mathbb{E}\left[|R_{j,1}(f)|\right] \leq \sum_{j=1}^{\infty} j\mathbb{E}\left[R_{j,1}^2(f)\right] < \infty \tag{4.3.25}$$

and $\Psi_j(N) \leq 1$ for all $1 \leq j \leq N$. Obviously, by Lemma 4.2.3,

$$\lim_{N\to\infty} \mathbb{E}\left[|G_{\sigma,N,z} - H_{N,z}|\right] = 0 \tag{4.3.26}$$

holds. This shows that

$$G_{\sigma,N,z} - \mathbb{E}\left[G_{\sigma,N,z}\right] \tag{4.3.27}$$

and

$$H_{N,z} - \mathbb{E}\left[H_{N,z}\right] \tag{4.3.28}$$

have the same limit distribution. If $\theta < 1$, it is well-known that if the sequence $\left(j\mathbb{E}\left[|\hat{R}_j(f)|\right]\right)_{j\geq 1}$ converges in Cesaro (C,θ) to zero, then the sequence converges also in Cesaro $(C,1)$ (see Lemma 2.2, [10]). Therefore, (4.3.26) holds has well by Lemma 4.2.7. So,

$$\mathbb{E}\left[G_{\sigma,N,z}\right] = \mathbb{E}\left[H_{N,z}\right] + o(1). \tag{4.3.29}$$

But by definition, the distribution of $G_{\sigma,N,z} - \mathbb{E}\left[G_{\sigma,N,z}\right]$ has the same distribution as

$$I_{\sigma,N,z} - \mathbb{E}\left[I_{\sigma,N,z}\right]. \tag{4.3.30}$$

Then, Lemma 4.3.2 and Lemma 4.3.3 prove the first point in Theorem 4.3.1.
For the second point in Theorem 4.3.1, note that by (4.3.29),

$$\mathbb{E}\left[I_{\sigma,N,z}\right] = N\int_0^1 f(x)dx + \mathbb{E}\left[G_{\sigma,N,z}\right] = N\int_0^1 f(x)dx + \mathbb{E}\left[H_{N,z}\right] + o(1). \tag{4.3.31}$$

By (4.3.7), this proves (2) of Theorem 4.3.1.
If one assumes additionally that $\left(j^2\mathbb{E}\left[\hat{R}_j^2(f)\right]\right)_{j\geq 1}$ converges in Cesaro $(C,\theta\wedge 1)$ to zero, then by Lemma 4.2.7

$$\lim_{N\to\infty} \mathbb{E}\left[(G_{\sigma,N,z} - H_{N,z})^2\right] = 0 \tag{4.3.32}$$

and

$$\operatorname{Var} I_{\sigma,N,z} = \operatorname{Var} G_{\sigma,N,z} = \operatorname{Var} H_{N,z} + o(1). \tag{4.3.33}$$

The computation of the variance of $H_{N,z}$ in (4.3.8) proves the third point and completes the proof of Theorem 4.3.1. □

We continue by showing that Theorem 4.3.1 implies Theorem 4.1.1. For this, we state the a Lemma on functions of bounded variations.
Recall that

$$jR_j(\tau_{j,\ell}f) = jR_{j,\ell}(f) = \sum_{k=0}^{j-1} \tau_{j,\ell}f\left(\frac{k}{j}\right) - j\int_0^1 \tau_{j,\ell}f(x)dx, \tag{4.3.34}$$

where τ denotes the translation of f by a random term ψ_j^ℓ, i.i.d. for all ℓ when j is fixed, i.e.

$$\tau_{j,\ell}f(x) = f(x+\psi_j^{(\ell)}). \tag{4.3.35}$$

Lemma 4.3.4. *Let f be of bounded variation and write $TV(f)$ for the total variation of f. Then*

$$|R_{j,\ell}(f)| \leq \frac{TV(f)}{j} \tag{4.3.36}$$

Proof. Assume f is a positive function.

$$jR_{j,\ell}(f) = \sum_{k=0}^{j-1} \left[f(\frac{k}{j} + \psi_j^\ell) - j\int_{k/j}^{k+1/j} f(x+\psi_j^\ell)dx \right]. \tag{4.3.37}$$

Since f is a periodic function on $[0,1]$,

$$j\int_{k/j}^{k+1/j} f(x+\psi_j^\ell)dx \leq \max_{\tau_{j,\ell}} [\tau_{j,\ell} f(k+1/j), \tau_{j,\ell} f(k/j)] \tag{4.3.38}$$

and

$$j\int_{k/j}^{k+1/j} f(x+\psi_j^\ell)dx \geq \min_{\tau_{j,\ell}} [\tau_{j,\ell} f(k+1/j), \tau_{j,\ell} f(k/j)]. \tag{4.3.39}$$

Therefore,

$$|jR_{j,\ell}(f)| \leq \sum_{k=0}^{j-1} \left[\max_{\tau_{j,\ell}} |\tau_{j,\ell} f(k/j) - \tau_{j,\ell} f(k+1/j)| \right] \leq TV(f). \tag{4.3.40}$$

Since f is of bounded variation, f can be written as $f = f^+ + f^-$ and using the same arguments gives the result.

□

Hence, the arguments from [10] apply in our case.

Proof of Theorem 4.1.1. For $\theta \geq 1$, it suffices to assume condition (4.1.14) in order that the conclusions of Theorem 4.1.1 hold, since (4.3.25) implies (4.3.26). The condition of f being of bounded variation is needed whenever $\theta < 1$. But if f is of bounded variation, so is any translation of f by Lemma 4.3.4. So, by Lemma 2.2 [10], the bounded sequence $(j \sup_{j,z} \hat{R}_j(f))_{j\geq 1}$ converges in (C,θ) to zero, for any $\theta > 0$. Also, by Lemma 5.2 (iii) [10], the sequence $(j^2 \sup_{j,z})\hat{R}_j^2(f))_{j\geq 1}$ converges in (C,θ) to zero, for any $\theta > 0$. This completes the proof of Theorem 4.1.1. □

4.4 Gaussian Limit

In this section, we will give a proof of Theorem 4.1.5. In particular, we will prove that Theorem 4.1.5 is implied by the following theorem:

Theorem 4.4.1. *Assume that*

$$\sum_{j=1}^{\infty} \theta j \mathbb{E}\left[\hat{R}_j^2\right] = \infty \tag{4.4.1}$$

and

$$\sup_{j,z} |j\hat{R}_j| = o(\sqrt{\operatorname{Var} H_{N,z}}). \tag{4.4.2}$$

Then,

$$\frac{H_{N,z} - \mathbb{E}\left[H_{N,z}\right]}{\sqrt{\operatorname{Var} H_{N,z}}} \overset{d}{\Rightarrow} \mathcal{N}(0,1) \tag{4.4.3}$$

Let us denote

$$\eta_N^2 = \operatorname{Var} H_{N,z} \tag{4.4.4}$$

We give the proof by the following two lemmas.

Lemma 4.4.2. *Let ν_j be the distribution of $jR_{j,1}(f)$ and m_j its expectation. Then, the distribution μ_N of*

$$\widetilde{H}_{N,z} = \frac{H_{N,z} - \mathbb{E}\left[H_{N,z}\right]}{\eta_N} \tag{4.4.5}$$

satisfies

$$\log \hat{\mu}_N(t) = \sum_{j=1}^{N} \frac{\theta}{j}\left(\hat{\nu}_j(t/\eta_N) - \frac{itm_j}{\eta_N} - 1\right). \tag{4.4.6}$$

Proof. Let $V_j = \sum_{\ell=1}^{Y_j} jR_{j,\ell}(f)$ and ν_j being the distribution of $jR_{j,1}$, then

$$\begin{aligned}
\log \mathbb{E}\left[e^{it\widetilde{H}_{N,z}}\right] &= \log \mathbb{E}\left[e^{it/\eta_N(\sum_{j=1}^N V_j - m_j\theta/j)}\right] = \log \mathbb{E}\left[\prod_{j=1}^N e^{it/\eta_N(V_j - m_j\theta/j)}\right] && (4.4.7)\\
&= \log\left(\prod_{j=1}^N e^{-it/\eta_N\cdot m_j\theta/j}\,\mathbb{E}\left[e^{it/\eta_N\cdot V_j}\right]\right), && (4.4.8)
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[e^{it/\eta_N\cdot V_j}\right] &= \mathbb{E}\left[e^{it/\eta_N\cdot\sum_{\ell=1}^{Y_j} jR_{j,\ell}}\right] = \mathbb{E}\left[(\hat{\nu}_j(t/\eta_N))^{Y_j}\right] && (4.4.9)\\
&= \exp\left(\theta/j(\hat{\nu}_j(t/\eta_N) - 1)\right). && (4.4.10)
\end{aligned}$$

This gives

$$\begin{aligned}
\log \mathbb{E}\left[e^{it\widetilde{H}_{N,z}}\right] &= \log\left(\prod_{j=1}^N e^{-it/\eta_N\cdot m_j\theta/j} e^{\theta/j(\hat{\nu}_j(t/\eta_N)-1))}\right)\\
&= \sum_{j=1}^N \frac{\theta}{j}\left(\hat{\nu}_j(t/\eta_N) - \frac{it}{\eta_N}m_j - 1\right). && (4.4.11)
\end{aligned}$$

□

Lemma 4.4.3. *Assume that*

$$\sum_{j=1}^{\infty} \theta j \mathbb{E}\left[R_{j,1}^2\right] = \infty \tag{4.4.12}$$

and

$$\sup_{j,z} |jR_{j,1}| = o(\eta_N). \tag{4.4.13}$$

Then the distribution μ_N of $\widetilde{H}_{N,z}$ converges in distribution to $\mathcal{N}(0,1)$.

Proof. We proof the convergence in distribution by the Lévy-Khintchine Convergence Theorem. Recall that a distribution μ is infinitely divisible if its Fourier transform admits the following (unique) presentation in terms of (a, M, σ^2):

$$\widehat{\mu}(t) = \exp\left(\int \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x) + iat - \frac{1}{2}\sigma^2 t^2\right) \tag{4.4.14}$$

for an admissible measure M.
Let $\Gamma_{j,N}$ be the distribution of $jR_{j,1}/\eta_N$. The distribution μ_N of $\widetilde{H}_{N,z}$ has then a Lévy-Khintchine representation in terms of (a_N, M_N, σ_N^2), where

$$M_N = \sum_{j=1}^{N} \frac{\theta}{j} \Gamma_{j,N}, \quad \sigma_N^2 = 0 \tag{4.4.15}$$

and

$$a_N = \frac{1}{\eta_N} \sum_{j=1}^{N} \frac{\theta}{j} \left(\mathbb{E}\left[\frac{jR_{j,1}}{1 + j^2 R_{j,1}^2/\eta_N^2}\right] - m_j\right). \tag{4.4.16}$$

For every N,

$$\int x^2 dM_N(x) = \sum_{j=1}^{N} \frac{\theta}{j} \mathbb{E}\left[\frac{j^2 R_{j,1}^2}{\eta_N^2}\right] = \frac{1}{\eta_N^2} \sum_{j=1}^{N} \theta j \mathbb{E}\left[R_{j,1}^2\right] = 1 \tag{4.4.17}$$

and therefore M_N and M are admissible measures with

$$M = \lim_{N\to\infty} \sum_{j=1}^{N} \frac{\theta}{j} \Gamma_{j,N}. \tag{4.4.18}$$

Moreover, for any function f being bounded and continuous such that $f(x) = 0$ for $|x| > \delta$,

$$\int f(x) dM_N(x) = \sum_{j=1}^{N} \frac{\theta}{j} \mathbb{E}\left[f\left(\frac{jR_{j,1}}{\eta_N}\right)\right], \tag{4.4.19}$$

so that we have by assumption (4.4.13)

$$\lim_{N\to\infty} \int f(x) dM_N(x) = \int f(x) dM(x) = 0 \tag{4.4.20}$$

For any $\ell > 0$,

$$\int_{-\ell}^{\ell} x^2 M_N(x) + \sigma_N^2 = \frac{1}{\eta_N^2} \sum_{j=1}^{N} \theta j \mathbb{E}\left[R_{j,1}^2\right] \mathbb{1}_{|jR_{j,1}|<\ell\eta_N}. \tag{4.4.21}$$

So,

$$\lim_{N\to\infty} \int_{-\ell}^{\ell} x^2 M_N(x) + \sigma_N^2 = \lim_{N\to\infty} \frac{1}{\eta_N^2} \sum_{j=1}^{N} \theta j \mathbb{E}\left[R_{j,1}^2\right] = 1 = \sigma^2 \tag{4.4.22}$$

Then it is left to show that $|a_N| \to 0$.

$$\begin{aligned} a_N &= \frac{1}{\eta_N} \sum_{j=1}^{N} \frac{\theta}{j} \left(\mathbb{E}\left[\frac{jR_{j,1}}{1 + j^2 R_{j,1}^2/\eta_N^2} \right] - m_j \right) & (4.4.23) \\ &= \frac{1}{\eta_N} \sum_{j=1}^{N} \frac{\theta}{j} \left(\mathbb{E}\left[\frac{jR_{j,1}\eta_N^2}{\eta_N^2 + j^2 R_{j,1}^2} - \frac{jR_{j,1}\eta_N^2}{\eta_N^2 + j^2 R_{j,1}^2} - \frac{j^3 R_{j,1}^3}{\eta_N^2 + j^2 R_{j,2}^2} \right] \right) & (4.4.24) \end{aligned}$$

So,

$$\begin{aligned} |a_N| &\leq \frac{1}{\eta_N} \sum_{j=1}^{N} \frac{\theta}{j} \mathbb{E}\left[\frac{j^3 R_{j,1}^3}{\eta_N^2 + j^2 R_{j,2}^2} \right] \leq \frac{1}{\eta_N} \sum_{j=1}^{N} \frac{\theta}{j} \mathbb{E}\left[\frac{j^3 R_{j,1}^3}{\eta_N^2} \right] & (4.4.25) \\ &\leq \frac{\sup_{j,z} |jR_{j,1}|}{\eta_N^3} \sum_{j=1}^{N} \frac{\theta}{j} \mathbb{E}\left[j^2 R_{j,1}^2\right] & (4.4.26) \\ &= \frac{\sup_{j,z} |jR_{j,1}|}{\eta_N}. & (4.4.27) \end{aligned}$$

By assumption (4.4.13), we see that

$$\lim_{N\to\infty} a_N = 0. \tag{4.4.28}$$

By (4.4.20), (4.4.22), (4.4.28) and by applying the Lévy-Khintchine Convergence Theorem (see for example Theorem 3.21, p.62 [74]), we see that μ_N converges in law to $\mathcal{N}(0,1)$ with Lévy-Khintchine representation $(a, M, \sigma) = (0, 0, 1)$. ☐

Proof of Theorem 4.4.1 . Theorem 4.4.1 follows immediately by Lemma 4.4.2 and Lemma 4.4.3. ☐

We continue by using the Feller Coupling to prove the following:

Lemma 4.4.4. *Under the assumptions that*

$$\lim_{N\to\infty} \eta_N^2 = \infty \tag{4.4.29}$$

and

$$\sup_{j,z} |jR_{j,1}(f)| = o(\eta_N), \tag{4.4.30}$$

$$\frac{I_{\sigma,N,z}(f) - \mathbb{E}\left[I_{\sigma,N,z}(f)\right]}{\sqrt{\operatorname{Var} I_{\sigma,N,z}(f)}} \overset{d}{\Rightarrow} \mathcal{N}(0,1) \tag{4.4.31}$$

Proof. It is clear that showing

$$\frac{G_{\sigma,N,z} - \mathbb{E}\left[G_{\sigma,N,z}\right]}{\sqrt{\operatorname{Var} G_{\sigma,N,z}}} \overset{d}{\Rightarrow} \mathcal{N}(0,1) \tag{4.4.32}$$

proves the claim.
Define

$$\widetilde{G}_{\sigma,N,z} = \frac{G_{\sigma,N,z} - \mathbb{E}\left[G_{\sigma,N,z}\right]}{\eta_N}. \tag{4.4.33}$$

By Lemma 4.2.5 and the assumption that $\sup_{j,z} jR_{j,1}(f) = o(\eta_N)$,

$$\mathbb{E}\left[|\widetilde{G}_{\sigma,N,z} - \widetilde{H}_{N,z}|\right] \le \frac{1}{\eta_N} 2\mathbb{E}\left[|G_{\sigma,N,z} - H_{N,z}|\right] = o(1), \tag{4.4.34}$$

which shows that

$$\mathbb{E}\left[G_{\sigma,N,z}\right] = \mathbb{E}\left[H_{N,z}\right] + o(\eta_N). \tag{4.4.35}$$

Moreover,

$$\mathbb{E}\left[(\widetilde{G}_{\sigma,N,z} - \widetilde{H}_{N,z})^2\right] \le 2\left(\mathbb{E}\left[(\widetilde{G}_{\sigma,N,z} - \widetilde{H}_{N,z})^2\right] + \mathbb{E}\left[|\widetilde{G}_{\sigma,N,z} - \widetilde{H}_{N,z}|\right]^2\right) = o(\eta_N^2) \tag{4.4.36}$$

So,

$$\operatorname{Var} G_{\sigma,N,z} = \operatorname{Var} H_{N,z} + o(\eta_N^2) \tag{4.4.37}$$

and by Lemma 4.4.1,

$$\frac{G_{\sigma,N,z} - \mathbb{E}\left[G_{\sigma,N,z}\right]}{\sqrt{\operatorname{Var} G_{\sigma,N,z}}} \overset{d}{\Rightarrow} \mathcal{N}(0,1). \tag{4.4.38}$$

□

Now, we will give the Proof of Theorem 4.1.5.

Proof of Theorem 4.1.5. Assume that $\lim_{N\to\infty} \eta_N^2 = \infty$. If f is of bounded variation, then by Lemma 4.3.4 $jR_{j,\ell}(f) = O(1)$ and the assumptions in Lemma 4.4.4 are satisfied, which proves the first statement in Theorem 4.1.5. Moreover, by Lemma 4.2.5,

$$\begin{aligned}
\mathbb{E}\left[I_{\sigma,N,z}\right] &= N\int_0^1 f(x)dx + \mathbb{E}\left[G_{\sigma,N,z}\right] && (4.4.39)\\
&= N\int_0^1 f(x)dx + \mathbb{E}\left[H_{N,z}\right] + O(1) && (4.4.40)\\
&= N\int_0^1 f(x)dx + \sum_{j=1}^{N} \theta\mathbb{E}\left[R_{j,1}(f)\right] + O(1) && (4.4.41)
\end{aligned}$$

and

$$\operatorname{Var} I_{\sigma,N,z} = \operatorname{Var} G_{\sigma,N,z} = \operatorname{Var} H_{N,z} + o(\eta_N^2), \tag{4.4.42}$$

which shows that

$$\operatorname{Var} G_{\sigma,N,z} \sim \operatorname{Var} H_{N,z} = \eta_N^2. \tag{4.4.43}$$

This concludes the proof of Theorem 4.1.5. □

We discuss one simple case where the assumptions $\lim_{N\to\infty}\eta_N^2 = \infty$ and $\sup_{j,z}|jR_{j,1}(f)| = o(\eta_N)$ are satisfied:

Remark 4.4.4.1. *Consider the indicator function on an interval* $(a,b]$ *on* $[0,1]$ *(see [76]). Then*

$$\mathbb{E}\left[I_{\sigma,N,z}\right] = N(b-a) + \sum_{j=1}^{N}\frac{\theta}{j}\mathbb{E}\left[\{j(a-\psi_j)\} - \{j(b-\psi_j)\}\right] + O(1), \tag{4.4.44}$$

where $\{x\}$ *denotes the fractional part of* x*. From [76], it is known that if* $\psi_j \equiv 0$*, then*

$$\sum_{j=1}^{N}\frac{\theta}{j}(\{ja\} - \{jb\}) = C(a,b)\log N, \tag{4.4.45}$$

and the constant $C(a,b)$ *depends on if* a *and* b *are rational or not. Of course, if* ψ_j *is not zero, then* $C(a,b)$ *becomes a random variable and its expectation depends on* $\mathbb{E}\left[\{j(a-\psi_j)\} - \{j(b-\psi_j)\}\right]$ *being rational or not.*

Remark 4.4.4.2. *The conditions of Theorem 4.1.5 are slightly different from a similar result by Hughes, Najnudel, Nikeghbali and Zeindler [43], where they prove a CLT conditioned on a p-Norm, instead of the* sup*-Norm.*

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